(31) $3^2\cdot5\cdot13\begin{cases}11\cdot199\\29\cdot79\end{cases}$	(54) $3^2\cdot5^2\begin{cases}11\cdot59\cdot179\\17\cdot19\cdot359\end{cases}$	(13) $3^2\cdot5\cdot13\cdot19\begin{cases}29\cdot569\\17099\end{cases}$
(33) $3^2\cdot5\cdot13\cdot19\begin{cases}37\cdot1583\\227\cdot263\end{cases}$	(32) $3^2\cdot5\cdot13\begin{cases}19\cdot47\\29\cdot31\end{cases}$	(12) $3^2\cdot7^2\cdot11\cdot13\begin{cases}41\cdot461\\19403\end{cases}$
(41) $3^2\cdot7\cdot13\cdot23\begin{cases}11\cdot19\cdot367\\79\cdot1103\end{cases}$	(34) $3^2\cdot7^2\cdot13\cdot19\begin{cases}11\cdot220499\\89\cdot29399\end{cases}$	(false)
(30) $3^3\cdot5\begin{cases}7\cdot71\\17\cdot31\end{cases}$	(55) $3^2\cdot5\begin{cases}7\cdot21491\\17\cdot23\cdot397\end{cases}$	(42) $3^3\cdot5\cdot23\begin{cases}11\cdot19\cdot367\\79\cdot1103\end{cases}$
(11) $3^4\cdot5\cdot11\begin{cases}29\cdot89\\2699\end{cases}$	(56) $3^4\cdot7\cdot11^2\cdot19\begin{cases}47\cdot7019\\389\cdot863\end{cases}$	(57) $3^4\cdot7\cdot11^2\cdot19\begin{cases}53\cdot6959\\179\cdot2087\end{cases}$
(53) $3^5\cdot7^2\cdot13\cdot53\begin{cases}11\cdot211\\2543\end{cases}$	(58) $3^5\cdot7^2\cdot13\cdot19\begin{cases}47\cdot7019\\389\cdot863\end{cases}$	(59) $3^6\cdot7^2\cdot13\cdot19\begin{cases}53\cdot6959\\179\cdot2087\end{cases}$

Euler's final list of 61 pairs did not include the pairs α, β, γ, although he had obtained α four times in the body of his paper, viz., in (2), (3_1), (5_3); β twice in (3_1); γ in (2). Moreover, these three unlisted pairs occur as VIII, IX, and XIII among the 30 pairs in Euler's[362] earlier list, a fact noted on p. XXVI and p. LVIII of the Preface by P. H. Fuss and N. Fuss to Euler's Comm. Arith. Coll., who failed to observe that these three pairs occur in the text of Euler's present paper. Nor did these editors note that the fourth mentioned case of divergence between the two lists is due merely to the misprint[364c] of 57 for 47 in (43) of the present list, so that the correctly printed pair XXVIII of the list of 30 is really this (43) and not a new pair, as supposed by them.

From the fact that Euler obtained in his posthumous tract[97] on amicable numbers the pairs α, β (once on p. 631 and again on p. 633 and finally on p. 635), the editors inferred, p. LXXXI of the Preface, that the tract differs in analysis from the long paper just discussed. But no new pairs are found, while the cases treated on pp. 631–2 are merely problems 1 and 2 of Euler's preceding paper. It is different with p. 634, where Euler started with two numbers like 71 and 5·11 which, by his table, have the same sum, 72, of divisors, and required a number a relatively prime to them such that $71a$ and $55a$ are amicable. The single condition is $72\int a=(71+55)a$, whence $\int a:a=7:4$. Thus a has the factor 4. If $a=4b$, where b is odd, then $\int b=b=1$, and the pair 284, 220 results. The case $a=8b$ is impossible. This method was used in a special way by Kraft[363] who limited the numbers from which one starts to a prime and a product of two primes.

In the Encyclopédie Sc. Math., I, 3_1, p. 59, note 320, it is stated that this posthumous tract contains four pairs not in Euler's list of 61, two pairs being those of Fermat[354] and Descartes.[355] But these were listed as (2) and (3) by Euler and were obtained by him in case (1_1) and attributed to Descartes.

E. Waring[365] noted that 2^nx, 2^nyz are amicable if

$$x=\frac{2^nyz-2^{n+1}+1}{2^n-1},\qquad z=2^n-1+\frac{2^{2n}}{y-2^n+1},$$

where x, y, z are primes and $y-2^n+1$ divides 2^{2n}. He cited the first two such pairs of amicable numbers.

[364c] G. Eneström, Bibliotheca Math., (3), 9, 1909, 263.
[365] Meditationes algebraicae, 1770, 201; ed. 3, 1782, 342–3.

The first three pairs were given in an anonymous work.[366]

In 1796, J. P. Grüson[100] (p. 87) gave the usual rule (1) leading to the three first known amicable pairs (verwandte Zahlen).

A. M. Legendre[367] attributed the rule (1) to Descartes.

G. S. Klügel[368] gave a process leading to the choice of P and Q, left arbitrary by Kraft.[363] We have $A:a=R+1:PQ+R=2R-P-Q$. Thus $P+Q=\{R(2A-a)-a\}/A$, while PQ is given by Kraft's second equation. Hence P and Q are the roots of a quadratic equation. For example, if $A=4$, then

$$8P,\ 8Q=R-7\pm\sqrt{R^2-62R-63}.$$

The positive root of $x^2-62x-63=0$ lies between 60 and 61. Thus we try primes $\geqq 61$ for R, such that $R-7$ is divisible by 8. The first available R is 71, giving $P=11$, $Q=5$ and the amicable pair 220, 284. In general, the quantity $\alpha^2R^2+2\beta R+\gamma$ under the radical sign can be made equal to the square of $\alpha R+p$ (p arbitrary) by choice of R.

John Gough[369] considered amicable numbers ax, ayz, where x, y, z are distinct primes not dividing a. Let q be the sum of the aliquot divisors of a. Then

$$a+q+qx=ayz,\qquad x+1=(y+1)(z+1).$$

If $q\leqq a/4$, the first gives $ayz<(1+x)a/4$, while $2y\cdot2z>x+1$ by the second, Thus $q>a/4$. Let $a=r^n$, where r is a prime >1. Then $q=(a-1)/(r-1)$, which with $q>a/4$ implies $a(5-r)>4$, $r=2$ or 3. He proved that $r\neq3$. whence $r=2$, the case treated by van Schooten.[359]

J. Struve[103] cited his Osterprogramm, 1815, on amicable numbers.

A. M. Legendre[370] discussed the amicable numbers of the type (1_1) of Euler[364] (with Euler's m, k replaced by $m-\mu, \mu$). Legendre noted that $r=2^{2m+k}(2^k+1)^2-1$ is of the form s^2-1 and hence composite, if k is even; also that, if $k=3$, $p=9\cdot2^{m+3}-1$, $q=9\cdot2^m-1$, one of which is of the form s^2-1. He considered the new case $k=7$ and found for $m=1$ that $p=33023$, $q=257$, $r=8520191$, stating that if r be a prime we have the amicable numbers 2^8pq, 2^8r. This is in fact the case.[371] For $k=1$, we have the ancient rule (1); he proved that for $n\leqq15$ it gives only the known three pairs of amicable numbers.

Paganini[372], at age 16, announced the amicable numbers $1184=2^5.37$, $1210=2.5.11^2$, not in the list by Euler[364], but gave no indication of the method of discovery.

[366] Encyclopédie méthodique...Amusemens des Sciences Math. et Phys., nouv. éd., Padoue, 1793, I, 116. Cf. Les amusemens math., Lille, 1749, 315.
[367] Théorie des nombres, 1798, 463.
[368] Math. Wörterbuch, 1, 1803, 246–252 [5, 1831, 55].
[369] New Series of the Math. Repository (ed., Th. Leybourn), vol. 2, pt. 2, 1807, 34–39. He cited Hutton's Math. Dict., article Amicable Numbers, taken from van Schooten[359].
[370] Théorie des nombres, ed. 3, 1830, II, §472, p. 150. German transl. by H. Maser, Leipzig, 1893, II, p. 145.
[371] Tchebychef, Jour. de Math., 16, 1851, 275; Werke, 1, 90. T. Pepin, Atti Acc. Pont. Nuovi Lincei, 48, 1889, 152–6. Kraitchik, Sphinx-Oedipe, 6, 1911, 92. Also by Lehmer's Factor Table or Table of Primes.
[372] B. Nicolò I. Paganini, Atti della R. Accad. Sc. Torino, 2, 1866–7, 362. Cf. Cremona's Ital. transl. of Baltzer's Mathematatik, pt. III.