Integrals, an Introduction to Analytic Number Theory

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ILAN VARDI: I got my Ph.D. in Number Theory from M.I.T. in 1982, as a student of Dorian Goldfeld. I then spent a year at the Institute for Advanced Study. I was an acting assistant professor at Stanford from 1983 to 1985. After realizing that not everybody cared about Kloosterman Sums, I learned how to use a computer and tried out some applied math. I'm now interested in special functions related to determinants of Laplacians.

1. Introduction. An examination of Gradzhteyn and Ryzhyk's book of integral tables reveals a large number of difficult and obscure integral formulas. In my opinion one of the most remarkable is given on p. 532

$$\int_{\pi/4}^{\pi/2} \log\log\tan x\,dx = \frac{\pi}{2}\log\left[\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right], \tag{1}$$

where

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt, \qquad s > 0$$

is the classical Γ-function. The reference given is to Bierens de Haan [**2**]. Failing to locate the proof of this formula, I decided to study equation (1) in some depth. It turns out that this formula requires some fairly involved analysis to prove, and also serves as a good example of how nontrivial number theory can be embedded in an integral formula.

The key to equation (1) is the *Dirichlet L-function*

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s}\cdots.$$

This is a well-known function; for example every calculus student knows the formula

$$L(1) = 1 - \frac{1}{3} + \frac{1}{5}\cdots = \frac{\pi}{4}.$$

Also, by the alternating series test $L(s)$ converges for $0 < s < 1$. However, much more is known and Hurwitz proved that $L(s)$ can be analytically continued to an entire function in the whole complex plane. He did this by proving the *functional equation*

$$L(1-s) = \left(\frac{2}{\pi}\right)^s \sin\frac{\pi s}{2}\Gamma(s)L(s). \tag{2}$$

What we will, in fact, show is that

$$\int_{\pi/4}^{\pi/2} \log\log\tan x\,dx = \frac{d}{ds}\Gamma(s)L(s)\Big|_{s=1} \tag{3}$$

Invoking the well-known formulas

$$\Gamma(1) = 1$$
$$\Gamma'(1) = -\gamma,$$

where γ is *Euler's constant*,

$$\gamma = \lim_{n\to\infty}\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n\right\} = .5772156649015328606065 12\ldots,$$

equation (3) becomes

$$\int_{\pi/4}^{\pi/2}\log\log\tan x\,dx = -\gamma\frac{\pi}{4} + L'(1).$$

So the proof of equation (1) will consist of 2 parts: a) establishing (3) b) expressing $L'(1)$ in terms of logarithms of Γ-functions.

2. Proof of equation (3). We begin with a general *Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

which, if f is of polynomial growth, will converge absolutely in a half-plane $Re(s) > c$. We now use the technique first developed by Riemann to study the Riemann ζ-function

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt = \int_0^\infty e^{-nt}(nt)^{s-1}\,d(nt),$$

so

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-nt}t^{s-1}\,dt.$$

Hence, by absolute convergence, one gets that for $Re(s) > c$

$$\Gamma(s)F(s) = \Gamma(s)\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \sum_{n=1}^{\infty}f(n)\int_0^\infty e^{-nt}t^{s-1}\,dt$$
$$= \int_0^\infty\left(\sum_{n=1}^{\infty}f(n)e^{-nt}\right)t^{s-1}\,dt.$$

Now let $z = e^{-t}$, this gives

$$\Gamma(s)F(s) = \int_0^1\left(\sum_{n=1}^{\infty}f(n)z^n\right)\left(\log\frac{1}{z}\right)^{s-1}\frac{dz}{z}.$$

Now we add the restriction that $f(n)$ be a *periodic* function. That is, there is a positive integer q such that $f(n+q)=f(n)$ for all n (for technical reasons also assume that $f(q)=0$). With these assumptions we have that for $|z|<1$

$$\sum_{n=1}^{\infty} f(n) z^n = \sum_{m=0}^{\infty} \sum_{n=1}^{q-1} f(mq+n) z^{mq+n}$$

$$= \frac{\sum_{n=1}^{q-1} f(n) z^n}{1-z^q} = \frac{P(z,f)}{1-z^q},$$

where

$$P(z,f) = \sum_{n=1}^{q-1} f(n) z^n.$$

We have thus obtained the formula:

$$F(s)\Gamma(s) = \int_0^1 \frac{P(z,f)\left(\log \frac{1}{z}\right)^{s-1}}{1-z^q} \frac{dz}{z}. \tag{4}$$

This formula was first obtained by Dirichlet (see [3]) to derive his *class number formula* of which $L(1)=\pi/4$ is the simplest case. Differentiating equation (4) by Leibniz's rule gives

$$\frac{d}{ds} F(s)\Gamma(s) = \int_0^1 P(z,f) \frac{\left(\log \frac{1}{z}\right)^{s-1}}{1-z^q} \log\log\left(\frac{1}{z}\right) \frac{dz}{z}.$$

Now if $F(s)$ converges absolutely at $s=1$ this will yield

$$F'(1) - \gamma F(1) = \int_0^1 P(z,f) \log\log\left(\frac{1}{z}\right) \frac{dz}{z}. \tag{5}$$

To prove equation (1) we let $q=4$ and pick $f(n)$ to be the *quadratic character* (mod 4) that is

$$\chi_4(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 1 & \text{if } n \equiv 1 \pmod 4 \\ 0 & \text{if } n \equiv 2 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$

χ_4 is called the quadratic character (mod 4) because for $(n,4)=1$

$$\chi_4(n) = \begin{cases} 1 & \text{if } \exists x \text{ s.t. } x^2 \equiv n \pmod 4 \\ -1 & \text{otherwise,} \end{cases}$$

while $\chi_4(n)=0$ if $(n,q)>1$.

So we have

$$P(z,\chi) = z - z^3$$

and equation (5) becomes

$$L'(1) - \gamma\frac{\pi}{4} = \int_0^1 \frac{(z - z^3)\log\log\frac{1}{z}}{1 - z^4}\frac{dz}{z}$$

$$= \int_0^1 \log\log\left(\frac{1}{z}\right)\frac{dz}{1 + z^2} = \int_1^\infty \log\log u \frac{du}{1 + u^2}$$

$$= \int_{\pi/4}^{\pi/2} \log\log\tan x\,dx.$$

3. Evaluating $L'(1)$. It turns out that it is much easier, first, to evaluate $L'(0)$, then use the functional equation $L(s) \to L(1 - s)$ to obtain the value for $L'(1)$.

To compute $L'(0)$ we follow a method due to André Weil [7]. Let

$$\zeta(s, a) = \sum_{n=1}^{\infty} \frac{1}{(n + a)^s}, \qquad 0 < a \le 1,$$

be the *Hurwitz* ζ*-function*. It is easily shown to converge for $Re(s) > 1$. Using the integral formula [8]

$$\Gamma(s)\zeta(a, s) = \int_0^\infty \frac{e^{-at}}{1 - e^t} t^{s-1}\,dt$$

one can show that $\zeta(a, s)$ can be analytically continued to the whole complex plane with only a simple pole at $s = 1$. The relevance of $\zeta(a, s)$ is due to the formula

$$L(s) = 4^{-s}\left[\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right)\right];$$

thus evaluating $\zeta'(0, a)$ will yield the value of $L'(0)$ (for ease of notation we have written $\zeta'(s, a)$ to denote $\frac{\partial}{\partial s}\zeta(s, a)$). Weil's observation is the following: note that for $s > 1$

$$\zeta(s, a + 1) = \zeta(s, a) - \frac{1}{a^s},$$

thus

$$\zeta'(s, a + 1) = \zeta(s, a) + a^{-s}\log a,$$

and at $s = 0$

$$\zeta(0, a + 1) = \zeta(0, a) + \log a.$$

Letting

$$G(a) = e^{\zeta'(0, a)},$$

we see that $G(a)$ satisfies the functional equation

$$G(a+1) = aG(a).$$

Further, one has that

$$\frac{d^2}{da^2}\log G(a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^2} > 0, \quad \text{for } a > 0,$$

and that

$$G(a) \text{ is analytic for } a > 0.$$

These however are the exact conditions for the *Bohr-Mollerup Theorem* for the uniqueness of the Gamma function [**1**]. Thus one has that

$$G(a) = G(1)\Gamma(a).$$

One sees that $G(1) = \zeta'(0,1)$, and on noting that $\zeta(s,1) = \zeta(s)$, where $\zeta(s)$ is the Riemann ζ-function, one has

$$G(1) = \zeta'(0).$$

It is well known that $\zeta'(0) = -(1/2)\log 2\pi$ (e.g., [**6**], [**8**]), and so

$$\zeta'(0,a) = \log\frac{\Gamma(a)}{\sqrt{2\pi}}.$$

Substituting this in the formula for $L(s)$ one derives

$$L'(0) = \log\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} - L(0)\log 4.$$

By the functional equation and $L(1) = \pi/4$ one gets that

$$L(0) = \frac{1}{2}.$$

And once again by the functional equation

$$\frac{2}{\pi}\frac{d}{ds}\Gamma(s)L(s)\Big|_{s=1} = \frac{1}{2}\log 4 + \log\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \frac{1}{2}\log\frac{\pi}{2},$$

and thus

$$L'(1) = \gamma\frac{\pi}{2} + \frac{\pi}{2}\log\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}.$$

This concludes the proof of equation (1).

4. More formulas! There are actually quite a number of identities in Gradzhteyn and Ryzhyk similar to (1). For example, there are

$$\int_0^1 \log\log\left(\frac{1}{x}\right)\frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}}\log\left[\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}(2\pi)^{1/3}\right], \quad \text{page 571} \qquad (6)$$

$$\int_0^1 \log\log\left(\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\log 2\pi - \log\Gamma\left(\frac{1}{6}\right)\right], \quad \text{page 572.} \qquad (7)$$

One sees that in equation (6) 3 plays the "key role" and in equation (7) 6 is the "magic number." To explain this one introduces *Dirichlet characters* (mod q)

$$\chi \text{ is a Dirichlet character } (\text{mod } q) \text{ if}$$

$$\begin{aligned} \chi(1) &= 1 & & \\ \chi(n+q) &= \chi(n) & & \forall n \\ \chi(n) &= 0 & & \text{if } (n,q) > 1 \\ \chi(mn) &= \chi(m)\chi(n) & & \forall m,n. \end{aligned}$$

The corresponding Dirichlet L-function is

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \qquad Re(s) > 0$$

and can be continued to an entire function if χ is not the *trivial character*

$$\chi_0(n) = 1 \quad \text{if } (n,q) = 1$$

Now the analogous character to χ_4 in equation (6) is the quadratic character (mod 3)

$$\chi_3(n) = \begin{cases} 0 & \text{if } n \equiv 0 \ (\text{mod } 3) \\ 1 & \text{if } n \equiv 1 \ (\text{mod } 3) \\ -1 & \text{if } n \equiv 2 \ (\text{mod } 3), \end{cases}$$

and in equation (7) the corresponding character is the quadratic character (mod 6) $\chi_6(n)$. Hence we have the L-functions

$$L(s,\chi_3) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} \cdots$$

$$L(s,\chi_6) = 1 - \frac{1}{5^s} + \frac{1}{7^s} \cdots.$$

The proofs of (6) and (7) are completely analogous to our proof of equation (1).

One can further explain how the numbers 3, 4, 6 play the key roles in our formulas. First rewrite equation (1) in the same form as (6) and (7)

$$\int_0^1 \log\log\left(\frac{1}{x}\right)\frac{dx}{1+x^2} = \frac{\pi}{2}\log\left[\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right].$$

Note that the solutions of x^2+1 are 4th roots of unity, i and $-i$, and one explains why $L(s,\chi_4)$ is involved by noting that it can be shown from the *Quadratic Reciprocity Theorem* that

$$\zeta_{\mathbf{Q}(i)}(s) = L(s,\chi_4)\zeta(s),$$

where $\zeta_{\mathbf{Q}(i)}(s)$ is the *Dedekind zeta function* of the field $\mathbf{Q}(i)$, and the classical definition (e.g., [**5**]) of the Dedekind ζ-function of the number field K is

$$\zeta_K(s) = \sum_{\substack{A\subseteq K\\ A \text{ ideal}}} \frac{1}{N(A)^s}.$$

Similarly, x^2+x+1 is the irreducible polynomial for the 3rd roots of unity, $-1/2 \pm \frac{\sqrt{-3}}{2}$, and, as above, $L(s,\chi_3)$ appears because

$$\zeta_{\mathbf{Q}(\sqrt{-3})}(s) = L(s,\chi_3)\zeta(s).$$

Similarly, x^2-x+1 gives the 6th roots of 1, so, as above, one expects $L(s,\chi_6)$ to play the central role.

5. Exercises.

1) Show that

$$\int_1^e \log(-\log\log y)\,dy = -\sum_{n=1}^{\infty}\frac{\log n}{n!} - \gamma e.$$

Hint: consider

$$L_f(s) = \sum_{n=1}^{\infty}\frac{1}{(n-1)!n^s}.$$

2) Find a similar formula for

$$\int_e^{e^e} \log(-\log\log\log y)\,dy.$$

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