(M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972, (6.1.45), p. 257), it follows that the top portion is less than \(|T/s| e^{-\pi T/2}\) when \(T\) is large. The bottom portion is similarly negligible. The integral along the left side of the rectangle is bounded above by
\[
\int_{-R}^{T} \left| s^{M-1/2-it} \frac{\Gamma(1/2 - M + it)}{1 + 2^{1/2-M-it}} \right| dt.
\]
Using the fact that
\[
|\Gamma(1/2 + it - M)| = \frac{\Gamma(1/2 + it)}{(1/2 - M + it)(3/2 - M + it) \cdots (-1/2 + it)} < \frac{\Gamma(1/2 + it)}{(M - 1)!},
\]
it follows that the integral along the left side of the rectangle is less than
\[
\frac{|s|^{M-1/2}}{(M - 1)!} \int_{-\infty}^{\infty} |\Gamma(1/2 + it)| dt.
\]
Hence, letting \(M \to \infty\), \(T \to \infty\), and \(R \to \infty\) independently, it follows that \(I\) is given by the sum of the residues at the poles of the integrand. Computing these residues gives
\[
I = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{s^k}{1 + 2^k} + g(s),
\]
where \(g(s)\) is the Fourier series in \(\log s/\log 2\) given by
\[
g(s) = \sum_{k=-\infty}^{\infty} \Gamma\left(\frac{-(2k + 1)\pi i}{\log 2}\right) \exp\left(\frac{(2k + 1)\pi i \log s}{\log 2}\right) = 2 \sum_{k=0}^{\infty} \text{Re} \left\{ \Gamma\left(\frac{-(2k + 1)\pi i}{\log 2}\right) \exp\left(\frac{(2k + 1)\pi i \log s}{\log 2}\right) \right\}.
\]
The exponential rate of decay of the gamma function on the imaginary axis ensures that (4) is rapidly convergent. From (1) and (2), we have \(sf(s) = 1 - 2I\), and hence (3) gives the stated result.

Solved also by D. A. Darling, A. Hildebrand & the proposer, J. H. Lindsey II, H. C. Morris, and WMC Problems Group.

**A Functional-Differential Equation**

10573 [1997, 168]. Proposed by Y.-F. S. Péttermann, University of Geneva, Geneva, Switzerland. Find a continuous function \(f: [0, \infty) \to [0, \infty)\) satisfying \(f(0) = 0\) and the functional differential equation \(f'(t) = 1/f(f(t))\) for \(t > 0\), and show that no other such function exists.

**Solution by the editors.** Trying a solution of the form \(f(t) = ct^\theta\), we see that \(f(t) = \phi^2 - \phi^1 \phi^{-1}\) is a solution, where \(\phi\) is the golden section \((1 + \sqrt{5})/2\). Note that the unique fixed point of \(f\) is at \(\phi\). Now suppose that \(g: [0, \infty) \to [0, \infty)\) is another such function. It follows from the functional equation that \(g\) is infinitely differentiable, increasing, concave downward, and \(g(t) > t\) for small enough \(t\). If \(g\) were bounded, say \(g(t) \leq M\), then \(g'(t) = 1/g(g(t)) \geq 1/M\), so (since \(g(0) = 0\) we would have \(g(t) \geq (1/M)t\) and could conclude that \(g\) is not bounded. Hence \(g\) is unbounded, and so \(g'(t) \to 0\) as \(t \to \infty\). Thus eventually \(g(t) < t\), and \(g\) has a unique fixed point, say at \(\psi\).

We claim that \(\psi = \phi\). If not, then we may take \(\psi < \phi\). Then \(g(\psi) = \psi < f(\psi)\). Let \(s\) be the supremum of \([0, \psi] \cap \{t : g(t) \geq f(t)\}\). Note that 0 is in this set so the supremum exists. Because \(f\) and \(g\) are continuous, we have \(f(s) = g(s)\). For \(s < t < \psi\) we have \(g(t) < f(t)\) and \(t < f(t) \leq f(\psi) < \phi\). If \(t \in (\psi, \psi']\) and \(f(t) \in (s, \psi]\), then \(g(g(t)) < g(f(t)) < f(f(t))\). If \(t \in (s, \psi]\) and \(f(t) > \psi\), then \(f(f(t)) > f(\psi) > \psi = 1998\)

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\[ g(g(\psi)) \geq g(g(t)) \text{.} \] Thus we have \( f(f(t)) > g(g(t)) \) whenever \( t \in (s, \psi) \), and hence \( g'(t) > f'(t) \). Integrating gives \( g(s) - f(s) \leq g(\psi) - f(\psi) < 0 \), but this contradicts \( f(s) = g(s) \). The argument for the case \( \psi > \phi \) is identical but with the roles of \( f \) and \( g \) reversed.

In summary, if \( 0 < t < \phi \), then \( t < g(t) < \phi \), and if \( \phi < t \), then \( \phi < g(t) < t \). The point \( \phi \) is an attracting fixed point of \( g \), and its basin of attraction is all of \((0, \infty)\).

Integrating the differential equation gives \( \int_0^{f(x)} f(t) \, dt = x \) and similarly for \( g \). Since \( f(\phi) = g(\phi) = \phi \), we obtain
\[
\int_\phi^{f(x)} f(t) \, dt = \int_\phi^{g(x)} g(t) \, dt.
\]

We now show that \( f = g \) on \([1, 2]\). Let \( C = \sup\{|f(t) - g(t)| : t \in [1, 2]\} \) with the supremum attained at \( x \in [1, 2] \). Since \( \phi \) is an attracting fixed point for both \( f \) and \( g \), \( f(x) \) and \( g(x) \) are also in \([1, 2]\). From (1),
\[
\int_\phi^{g(x)} (f(t) - g(t)) \, dt = \int_{f(x)}^{g(x)} f(t) \, dt.
\]

The integrand on the left is at most \( C \), and the integrand on the right is at least \( 1 \). Therefore \( C(g(x) - \phi) \geq |g(x) - f(x)| = C \). Since \( g(x) \in [1, 2] \), \( |g(x) - \phi| < 1 \). This gives a contradiction unless \( C = 0 \). Thus \( f = g \) on \([1, 2]\).

Now suppose \( f = g \) on an interval \([a, b] \supseteq [1, 2]\). We claim that \( f = g \) on the larger interval \([f^{-1}(a), f^{-1}(b)]\). Indeed, if \( x \in [f^{-1}(a), f^{-1}(b)] \), then \( f(x) \in [a, b] \) so \( \int_\phi^{f(x)} f(t) \, dt = \int_\phi^{g(x)} g(t) \, dt \), and from (1) we get \( 0 = \int_{f(x)}^{g(x)} g(t) \, dt \). But \( g \) is positive, so \( f(x) = g(x) \).

Finally, the intervals \([1, 2] \subseteq [f^{-1}(1), f^{-1}(2)] \subseteq [f^{-2}(1), f^{-2}(2)] \subseteq \cdots \) exhaust \((0, \infty)\), so we may conclude that \( f = g \) everywhere.

Solved also by the proposer.

**Closure, Complement, and Arbitrary Union**

10577 [1997, 169]. *Proposed by Mark Bowron and Stanley Rabinowitz, MathPro Press, Westford, MA.* It is well known that no more than 14 distinct sets can be obtained from one set in a topological space by repeatedly applying the operations of closure and complement in any order. Is there any bound on the number of sets that can be generated if we further allow arbitrary unions to be taken in addition to closures and complements?

**Solution by John Rickard, Advanced Telecommunications Modules Ltd., Cambridge, UK.**

No. Let \( A_n \) be the set of reals in the half-open interval \([0, 1)\) that have a binary expansion with exactly \( n \) 1’s. For example, \( A_0 = [0] \), and \( A_1 = [1/2, 1/4, 1/8, \ldots] \). Assuming the usual topology, the closure of \( A_n \) is the union \( A_0 \cup A_1 \cup \cdots \cup A_n \). For \( n \geq 0 \), let \( S_0 = \bigcup_{k=0}^{n} A_{2k} \) and \( S_{n+1} = \bigcup_{k=0}^{n} A_{2k+1} \).

The closure of \( S_n \) is \( A_0 \cup A_1 \cup \cdots \cup A_n \), so closure(\( S_n \)) = \( S_n \), since the \( A_i \) are mutually disjoint. Thus each of the sets \( S_0, \ldots, S_n \) can be generated from \( S_n \) under the operations of closure and set difference. The operation of set difference may in turn be defined in terms of union and complement using DeMorgan’s laws, so this example shows that there is no finite bound.

It is also possible to generate infinitely many sets from one set of reals. An example of such a set is \( S_0 \cup (2 + S_1) \cup (4 + S_2) \cup (6 + S_3) \cup \cdots \).

**Editorial comment.** Luke Pebbey noted that for a given positive integer \( n \), there is a topological space and set in the space such that exactly \( n \) sets can be generated using these three operations if and only if \( n = 2^k \) for some \( k \). Both Pebbey and the proposers noted that there