Lyapunov Stability of Ground States of Nonlinear Dispersive Evolution Equations

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Introduction

A solitary wave is a localized, finite energy solution of a nonlinear evolution equation. It results from a balance of dispersion and a "focusing" nonlinearity. Two fundamental equations in the theory of nonlinear waves that possess such solutions are the nonlinear Schrödinger equation (NLS) and the Korteweg-de Vries equation (KdV). NLS arises in the mathematical description of electromagnetic wave propagation through nonlinear media (see [1],[27],[28]). KdV arises in the study of waves in shallow water (see [27]).

In this paper we present a new proof of orbital stability of ground state solitary waves of the nonlinear Schrödinger equation

\[ i\phi_t + \Delta \phi + f(|\phi|^2)\phi = 0, \]

\[ (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \]

\[ \phi(x,0) = \phi_0(x) \in H^1, \]

for a general class of nonlinearities, f. For the special case \( f(|\phi|^2) = |\phi|^{2\sigma} \), our analysis implies stability in the "subcritical" case, \( \sigma < 2/N \), in dimensions \( N = 1 \) and \( N = 3 \).

We also prove the stability of the solitary wave for the generalized Korteweg-de Vries equation

\[ w_t + a(w) w_x + w_{xxx} = 0, \]

\[ (x,t) \in \mathbb{R} \times \mathbb{R}^+, \]

\[ w(x,0) = w_0(x) \in H^2, \]

for \( a(t) \) of a general class. In the special case \( a(w) = w^p \), we have stability when \( p < 4 \).

Some discussion of linearized stability for NLS appeared in [29]. The first rigorous result on nonlinear stability of ground states for NLS was obtained by T.
Cazenave [7]. He proved stability relative to small symmetric perturbations for the special nonlinearity \( f(|\phi|^2) = \log|\phi|^2 \), by compactness methods. This result was extended by T. Cazenave and P. L. Lions [8], using “concentration compactness”, to some class of nonlinearities and general small perturbations. Stability for KdV \((a(t) = t)\) was proved by T. B. Benjamin [3], using the Lyapunov method. Corrections to Benjamin’s proof and less restrictive hypotheses were made by J. Bona [6]. Our aim is to present a new stability proof, for a general class of nonlinearities, by the Lyapunov method for NLS and a generalization of the work on KdV to GKdV.

The Lyapunov method has been used in studies of stability of incompressible ideal fluid flows by V. I. Arnold [2], barotropic fluid flow by Holm et. al. [14], and stability of “kinks” in the one-dimensional nonlinear Klein-Gordon equation by D. B. Henry, J. F. Perez, and W. F. Wreszinski [13]. E. W. Laedke and K. H. Spatschek [16] have results on the stability of three-dimensional Langmuir solitary waves, and have independently obtained results for generalizations of KdV in [17]. They obtain condition (*) below.

We shall now discuss the notion of stability we use and the Lyapunov method. NLS has phase and translation symmetries and GKdV has translation symmetry, i.e., if \( \phi(x, t) \) solves NLS and \( w(x, t) \) solves GKdV, then \( e^{i\gamma}(\phi(x + x_0, t) \) solves NLS and \( w(x + x_0, t) \) solves GKdV for any \( x_0 \in \mathbb{R}^N \) and \( \gamma \in [0, 2\pi) \). By orbital stability we mean stability modulo these symmetries. To make this precise we define the orbit of a function \( \psi \) to be

\[
\mathcal{G}_{\psi} = \{ \psi(\cdot + x_0)e^{i\gamma} | (x_0, \gamma) \in \mathbb{R}^N \times [0, 2\pi) \} \quad \text{for NLS},
\]

\[
\mathcal{G}_{\psi} = \{ \psi(\cdot + x_0) | x_0 \in \mathbb{R} \} \quad \text{for GKdV}.
\]

A ground state is orbitally stable if initial data being near the ground state orbit implies that the solution at all later times remains near the ground state orbit. This type of stability is to be expected since for a typical nonlinear wave the phase, speed and amplitude are coupled.

We measure the deviation of the solution \( \phi(t) \) from \( \mathcal{G}_{\psi} \) using the following metric:

\[
[\rho E(\phi(t), \mathcal{G}_{\psi})]^2 = \inf \{ \| \nabla \phi(\cdot + x_0, t) e^{i\gamma} - \nabla \psi \|_{L^2}^2
\]

\[
+ E \| \phi(\cdot + x_0, t) e^{i\gamma} - \psi \|_{L^2}^2 \},
\]

where the infimum is taken over all \( x_0 \in \mathbb{R}^N \) and \( \gamma \in [0, 2\pi) \) for NLS and all \( x_0 \in \mathbb{R} \) with \( \gamma \equiv 0 \) for GKdV. Since the minimum in (0.2) is attained (Section 3), this defines \( x_0(t) \) and \( \gamma(t) \). This choice of \( x_0(t) \) and \( \gamma(t) \) is crucial to our analysis, and differs from the choice of \( x_0 \) and \( \gamma \) in [8].

Our proof ensures the existence of the “modulations” \( x_0(t) \) and \( \gamma(t) \), but does not give these functions explicitly. In [24] a system of coupled nonlinear ordinary differential equations (modulation equations) are derived for \( x_0(t) \) and
γ(t). It is proved that if \( x_0(t) \) and \( γ(t) \) evolve according to the modulation equations, then the modulated ground state is linearly stable. In our present nonlinear analysis, we use extensively the analysis of the linearized NLS operator presented in [24].

We shall consider a one-parameter family of ground states \( R(x; E) \), up to phase and translation symmetries. To prove stability we construct a Lyapunov functional \( \mathcal{E}[φ] = \mathcal{H}[φ] + E\mathcal{N}(φ) \). Here \( \mathcal{H} \) and \( \mathcal{N} \) are, respectively, the Hamiltonian and square integral functionals, which are conserved for NLS and GKdV. We find that, restricted to the manifold of functions having square integral equal to \( \int R^2(x, E) \, dx \), \( R(•, E) \) (modulo symmetries) is a local minimum of \( \mathcal{E}[•] \), provided

\[
(•) \quad \frac{d}{dE} \mathcal{N}[R(•, E)] = -E^{-1} \frac{d}{dE} \mathcal{H}[R(•, E)] > 0.
\]

Stability of \( R(x, E) \) under condition (•) follows. For the special case, \( f(φ^2) = |φ|^{2σ} \), (•) reduces to the condition: \( σ < 2/N \). For \( σ \geq 2/N \), ground states are not stable in the above sense (see [5],[23],[22]). However, ground states do play a stable role in the formation of singularities in the critical case \( σ = 2/N \) (see [26]).

Condition (•) is arrived at through a spectral analysis of the linearized (about the ground state) NLS and GKdV operators. In contrast, the compactness approach to stability (see [8]) centers about the fact that minimizing sequences of certain variational problems are precompact (modulo symmetries). Condition (•) is the analogue of the convexity condition obtained by Shatah [21] for the stability of standing waves of nonlinear Klein-Gordon equations.

Our stability proof requires the fact that all "zero modes" of the linearized operator, \( L_+ \), are generated by spatial translation invariance of the nonlinear problem. We have been able to prove this completely only for \( N = 1 \) and \( N = 3 \). This is the source of restriction on spatial dimension in Theorem 2. We conjecture that this property holds in any spatial dimension. As was seen in [24], this is connected with the question of uniqueness of the ground state which has been only partially resolved (see [9],[18]).

Regarding GKdV, our proof is considerably simpler than that presented for KdV in [3],[6] in the following way. Rather than using the explicit spectral representation of the linearized operator that is available in dimension one, we use general spectral properties of the linearized operator which we can derive from a variational principle. Thus, our methods have the benefit of being applicable to higher-dimensional problems, e.g., NLS for \( N > 1 \).

This article is structured as follows. In Section 1 we present a variational characterization of the ground state. In Section 2 we state, for the case of power nonlinearities, the stability theorem for NLS (Theorem 2), and outline its proof. Here it is seen that stability relies on a suitable lower bound on the second variation of an energy functional. This lower bound is proved in Section 3. We use the analysis of a constrained variational problem carried out in [24],[25]. In Section 4 we consider NLS with more general nonlinearities (Theorem 3). In Section 5 we outline a proof of the stability of the solitary wave of GKdV (Theorem 4).
We shall use the notations
\[ \|f\|_p = \left( \int |f(x)|^p \, dx \right)^{1/p}, \]
\[ \|f\| = \|f\|_2, \]
\[ (f, g) = \int f(x)g(x) \, dx, \]
\[ \|f\|_{H^s}^2 = \sum_{|\alpha| \leq s} \|D^\alpha f\|^2, \]
\[ H^s = \{ f : \|f\|_{H^s} < \infty \}. \]

1. Ground States of NLS

In this section we discuss properties of NLS that will be used in the stability analysis.

Let the nonlinearity \( f \) have the following properties:

\[(A1) \quad \int_0^s f(t^2) t \, dt - \frac{1}{2} s^2 > 0 \quad \text{for some } s > 0; \]
\[(A2) \quad -\infty \leq \lim_{s \to -\infty} \sup f(s^2) s^{-4/N} \leq 0; \]
\[(A3) \quad f(s^2) = o(1) \quad \text{as } s \to 0. \]

By the work of J. Ginibre and G. Velo [12], NLS has a unique global solution of class \( C([0, \infty); H^1) \). In the special case \( f(|\phi|^2) = |\phi|^{2\alpha} \), it suffices to require \( \sigma < 2/N \). Their proof uses that the nonlinearity \( f \) is not too strong, (A2), and the conserved integrals

\[(1.1) \quad \mathcal{N}(\phi) = \int |\phi(x, t)|^2 \, dx, \]

and

\[(1.2) \quad \mathcal{H}(\phi) = \int \left| \nabla \phi(x, t) \right|^2 - G\left( |\phi(x, t)|^2 \right) \, dx, \]

where \( G(x) = \int_0^x f(t) \, dt \), to obtain an a priori bound on \( \|\nabla \phi(t)\| \) which is used to continue local solutions to global solutions in time. We use the conserved integrals, \( \mathcal{H} \) and \( \mathcal{N} \), to construct a Lyapunov function.


\[(1.3) \quad \Delta u - Eu + f(|u|^2)u = 0, \quad E > 0, \]

has a positive, radial, smooth and exponentially decaying solution which we call a
ground state and denote by $R$. Therefore,

$$\psi(x, t) = R(x; E) e^{iEt}, \quad E > 0,$$

is a solitary wave solution of NLS. In our stability analysis we shall exploit the following variational characterization of a ground state solitary wave for the case $f(|\phi|^2) = |\phi|^{2\sigma}$ introduced and applied in [23], [24], [25].

For $u \in H^1$ we define the functional

$$J[u] = \|\nabla u\|^{\sigma N} \|u\|^{2+\sigma(2-N)}/\|u\|^{2\sigma+2}.$$

**Theorem 1.** For $0 < \sigma < 2/(N-2)$,

$$\alpha \equiv \inf_{u \in H^1} J[u]$$

is attained at a function $R$ with the following properties:

(i) $R$ is positive and is symmetric with respect to some origin of coordinates,

(ii) $R \in H^1 \cap C^\infty$,

(iii) $R$ is a solution of

$$\Delta R - R + R^{2\sigma+1} = 0.$$

Finally, we note that, for any $\lambda, \gamma \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$,

$$\psi_{\lambda}(x, t) = \lambda^{1/\sigma} R(\lambda(x + x_0)) \exp\{i(\lambda^2 t + \gamma)\}$$

is a ground state of NLS with $f(|\phi|^2) = |\phi|^{2\sigma}$.

**2. Stability Theorem and Outline of the Proof**

We now state our result for the case $f(|\phi|^2) = |\phi|^{2\sigma}$.

**Theorem 2.** Let $\sigma < 2/N$ with $N = 1$ or $N = 3$. Let $\phi(x, t)$ be the unique solution of NLS with initial data $\phi_0 \in H^1$. Then the ground state is orbitally stable, i.e., for any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$, such that if

$$\rho_E(\phi_0, \mathcal{G}_R) < \delta(\varepsilon),$$

then for all $t > 0$

$$\rho_E(\phi(t), \mathcal{G}_R) < \varepsilon.$$

We shall now outline the general method we employ. As a Lyapunov functional we use the following conserved energy integral:

$$\mathcal{S}[\phi] = \mathcal{H}[\phi] + E\mathcal{N}[\phi] = \int|\nabla \phi|^2 - G(|\phi|^2) + E|\phi|^2 \, dx.$$
\( \mathcal{E} \) will be estimated in terms of \( \rho_E \) and will be used to measure the deviation of \( \phi(x, t) \) from the ground state orbit.

We write

\[
\phi(x + x_0, t)e^{i\gamma} = R(x) + w \quad \text{and} \quad w = u + iv.
\]

Then,

\[
\Delta \mathcal{E} = \mathcal{E}[\phi(\cdot, \cdot, t)] - \mathcal{E}[R(\cdot)] = \mathcal{E}[(x + x_0, t)e^{i\gamma}] - \mathcal{E}[R(\cdot)] \quad \text{by conservation of } \mathcal{E},
\]

\[
\geq (L_+ u, u) + (L_- v, v) - C_1||w||_{L^2}^2 - C_2||w||_{L^4}^4 \quad \text{with } \theta > 0.
\]

Here,

\[
L_+ = -\Delta + 1 - (f(R^2) + 2R^2f'(R^2)) \quad \text{and} \quad L_- = -\Delta + 1 - f(R^2)
\]

are, respectively, the real and imaginary parts of the linearized NLS operator about the ground state. The inequality in (2.5) is arrived at as follows. First Taylor expand the previous line about \( R \). The first variation of \( \mathcal{E} \) at \( R \) vanishes by (1.7). The second variation is the quadratic functional in \( u \) and \( v \). The remaining \( O(|w|^3) \) terms can be estimated from below by an interpolation estimate of Nirenberg and Gagliardo (see for example [11]).

In Section 3 we show that if \( x_0 = x_0(t) \) and \( \gamma = \gamma(t) \) are chosen to minimize

\[
\| \nabla \phi(\cdot + x_0, t)e^{i\gamma} - \nabla R(\cdot) \|^2 + E\| \phi(\cdot + x_0, t)e^{i\gamma} - R(\cdot) \|^2,
\]

that is are chosen as prescribed by the \( \rho_E \)-metric, then

\[
(L_+ u, u) + (L_- v, v) \geq C_3||w||_{L^2}^2 - C_4||w||_{L^4}^4 - C_5||w||_{L^4}^4.
\]

Putting (2.5) and (2.8) together, we have for \( x_0 \) and \( \gamma \) that minimize (2.7)

\[
\Delta \mathcal{E} \geq g(\rho_E(\phi(t), \mathcal{E}_R)),
\]

where

\[
g(t) = ct^2(1 - at^\theta - bt^4) \quad \text{with } a, b, c, \theta > 0.
\]
To obtain (2.9) we use that, for \(x_0\) and \(\gamma\) as above,

\[
\sqrt{\min(E, 1)} \|w(t)\|_{H^1} \leq \rho_E(\phi(t), \mathcal{G}_R) \leq \sqrt{\max(E, 1)} \|w(t)\|_{H^1}.
\]

The essential properties of \(g\) are that \(g(0) = 0\) and \(g(t) > 0\) for \(0 < t \leq 1\).

The stability result can be derived from (2.9) as follows. Let \(\varepsilon > 0\) be sufficiently small. Then, by the continuity of \(\mathcal{G}\) in \(H^1\), there is a \(\delta(\varepsilon) > 0\) such that if

\[
\rho_E(\phi_0, \mathcal{G}_R) < \delta(\varepsilon),
\]

then

\[
\Delta\mathcal{G}(0) < g(\varepsilon).
\]

Since \(\Delta\mathcal{G}\) is constant in time, (2.9) implies that \(g(\rho_E(\phi(t), \mathcal{G}_R)) < g(\varepsilon)\) for all \(t > 0\). Therefore, since \(\rho_E(\phi(t), \mathcal{G}_R)\) is a continuous function of time,

\[
\rho_E(\phi(t), \mathcal{G}_R) < \varepsilon \quad \text{for all} \quad t > 0,
\]

i.e., the ground state is orbitally stable.

3. Constrained Variational Problems for \(L_+\) and \(L_-\)

To convert the outline of Section 2 into a proof of Theorem 2 we must show that if, at time \(t > 0\), \(x_0(t)\) and \(\gamma(t)\) minimize (2.7), then (2.8) holds. In this section we carry this out for the case \(f(|\phi|^2) = |\phi|^{2\sigma}\).

That the minimum is attained at finite values \(x_0\) and \(\gamma\) and that \(w(t)\) as defined in (2.4) has a continuous \(H^1\) norm can be proved using the methods presented in [6]. Minimization of (2.7) over \(x_0\) and \(\gamma\) implies

\[
\int R^{2\sigma}(x) \frac{\partial R(x)}{\partial x_j} u(x, t) \, dx = 0, \quad j = 1, \ldots, N,
\]

and

\[
\int R^{2\sigma+1}(x) v(x, t) \, dx = 0.
\]

Since \(L_- R = 0\) and \(R > 0\), \(R\) is the ground state of \(L_-\), which is nondegenerate. Therefore, \(L_-\) is a non-negative operator. If we study the infimum of \((L_- v, v)/(v, v)\) subject to (3.2), we find (see [24]) that if it is zero, it is attained at \(R\). But this violates (3.2), implying that the minimum is positive. Therefore, (3.2) implies

\[
(L_- v, v) \geq C'(v, v)
\]

for some \(C' > 0\). It follows easily from (3.3) that

\[
(L_- v, v) \geq C'' \|v\|_{H^1}^2.
\]
Now $L_+$ has exactly one negative eigenvalue (Lemma 4.2) and (3.1) is not sufficient to ensure positivity of $(L_+ u, u)$. A suitable lower bound on $(L_+ u, u)$ is obtainable by further requiring that the perturbed solution have the same square integral as the ground state, i.e.,

\[(3.5) \quad \int |\phi(x, t)|^2 dx = \int R^2(x) dx.\]

By (2.4) and (3.5),

\[(3.6) \quad (u, R) = -\frac{1}{2} [(u, u) + (v, v)].\]

Condition (3.5) will later be relaxed. Constraints (3.1) and (3.6) on $(L_+ u, u)$ and (3.4) ensure the lower estimate (2.8). To see this, the main step is the following.

**Proposition 3.1.**

\[\inf_{(f, R) = 0} (L_+ f, f) = 0 \quad \text{if} \quad \sigma \leq 2/N.\]

We sketch a proof that was given in [24]. A more general argument, not restricted to the nonlinearity $|\phi|^{2\sigma}$, is presented in Section 4. Since $R$ is an unconstrained minimum of the functional $J$, defined in (1.5), the second variation of $J$ about $R$ is non-negative. After calculating $\delta^2 J$, we find that on functions $f$, satisfying $(f, R) = 0$,

\[(L_+ f, f) + k^2 (\sigma N - 2)(f, \Delta R)^2 \geq 0.\]

Therefore, for $\sigma \leq 2/N$, $(L_+ f, f) \geq 0$. Since $L_+ \nabla R = 0$ and $(\nabla R, R) = 0$, the proposition follows.

The following property of the null space of $L_+$ has been proved only in dimensions $N = 1$ and $N = 3$ (see [24]), though we conjecture it for any $0 < \sigma < 2/(N - 2)$. This is the only place in our analysis where we must restrict $N$, the spatial dimension.

**Proposition 3.2.** Let $N = 1$ or $N = 3$. Then

\[\text{Kernel } (L_+) = \text{span} \left\{ \frac{\partial R}{\partial x_j} : j = 1, \ldots, N \right\}.\]

**Proposition 3.3.** Let $\sigma < 2/N$. There are constants, $D$, $D'$ and $D'' > 0$ such that if $u$ satisfies (3.1) and (3.6), then

\[(3.7) \quad (L_+ u, u) \geq D\|u\|_2^2 - D'\|\nabla w\|_2^2 - D''\|w\|_4^4.\]

Proof: Without loss of generality, we can take $(R, R) = 1$. We write

\[(3.8) \quad u = u_\parallel + u_\perp,\]
where
\( u_\parallel \equiv (u, R) R \)
(3.9)
\[ = -\frac{1}{2} [(u, u) + (v, v)] R, \]
and
\( u_\perp \equiv u - (u, R) R \)
(3.10)
\[ = u + \frac{1}{2} [(u, u) + (v, v)] R \]
by (3.6).

Now,
\[ (L_+ u, u) = (L_+ u_\parallel, u_\parallel) + 2(L_+ u_\parallel, u_\perp) + (L_+ u_\perp, u_\perp). \]
(3.11)
We first consider the functional \((L_+ u_\parallel, u_\parallel)/(u_\parallel, u_\parallel)\). By Proposition 3.1, it is non-negative. If we study its infimum subject to (3.1) (see [24]), we find that, provided \( \sigma < 2/N \), if it is zero it is attained at \( c \cdot \nabla R \) for some \( c \in \mathbb{R}^N \). But this violates (3.1). Therefore,
\[ (L_+ u_\perp, u_\perp) \geq d(u_\perp, u_\perp) \]
(3.12)
\[ = d[(u, u) - (u_\parallel, u_\parallel)] \]
\[ = d[(u, u) - \frac{1}{2} [(u, u) + (v, v)]^2], \quad d > 0. \]

Also, by (3.6),
\[ (L_+ u_\parallel, u_\parallel) = \frac{1}{2} (L_+ R, R)[(u, u) + (v, v)]^2. \]
(3.13)

Finally,
\[ (L_+ u_\perp, u_\parallel) = (u, R)(L_+ u_\perp, R) \]
(3.14)
\[ = -\frac{1}{2} [(u, u) + (v, v)](L_+ u_\perp, R) \]
\[ \geq -d' ||w||^2 ||\nabla w||. \]

The lower bound (3.7) now follows from (3.12)–(3.14).

Estimates (3.4) and (3.7) imply (2.8). We have thus proved that \( R \) is stable modulo adjustments in \( x_0 \) and \( \gamma \) relative to small perturbations which preserve the \( L^2 \) norm. To prove stability relative to general perturbations we observe that given \( \phi_0 \) satisfying (2.1), there is a \( \lambda \) for which
\[ \psi_\lambda(x, t) = \lambda^{1/\sigma} R(\lambda x) \exp{\{i\lambda^2 t\}} \]
(3.15)

\(^1\)For \( \sigma = 2/N \), the function \((1/\sigma)R + x \cdot \nabla R \) makes \((L_+ f, f)/(f, f)\) vanish and satisfies constraints (3.1) and \((f, R) = 0\).
is a ground state of NLS, has the same $L^2$ norm as $\phi_0$, and

\begin{equation}
\|\psi_\lambda - R\|_{H^1} < \frac{1}{2} \varepsilon.
\end{equation}

Therefore,

\begin{equation}
\|\phi(\cdot + x_0, t)e^{it\gamma} - R\|_{H^1} \leq \|\phi(\cdot + x_0, t)e^{it\gamma} - \psi_\lambda\|_{H^1} + \|\psi_\lambda - R\|_{H^1}.
\end{equation}

Minimization of (2.7) over $x_0$ and $\gamma$, use of the stability result for perturbations preserving the $L^2$ norm, and (2.11) yield

\begin{equation}
\rho_\varepsilon(\phi(t), \mathcal{G}_R) < \varepsilon \quad \text{for all} \quad t > 0.
\end{equation}

We remark that this scheme of first considering perturbations which leave the $L^2$ norm unchanged breaks down when $u = 2/N$, since then the $L^2$ norm is invariant under the scale change $\phi(x) \to \lambda^{1/\sigma} \phi(\lambda x)$. In fact, the ground state is not stable in the present sense for $\sigma \geq 2/N$ (see [5], [23], [22], [26]).

4. General Nonlinearities

In this section we discuss a general class of nonlinear interactions $f(|\phi|^2)$. In addition to (A1)–(A3) we assume:

Given $E > 0$, there is a unique ground state $R(x; E)$, i.e., $R > 0$,

\begin{enumerate}
\item[(B1)] $R \in H^1$. (Uniqueness of the ground state has been studied in [9], [18].)
\end{enumerate}

$L_+$ has a null space that is spanned by the functions $\partial R/\partial x_j$,

\begin{enumerate}
\item[(B2)] $j = 1, 2, \cdots, N$, i.e., all zero modes are generated by translation invariance of (1.3)
\end{enumerate}

Properties (B1) and (B2) are closely related. They hold for the special case $f(|\phi|^2) = |\phi|^{2\sigma}$ when $N = 1$ and $N = 3$ (see Proposition 4.2 and [24]).

We also require

\begin{enumerate}
\item[(B3)] $\int [f(R^2) + 2R^2f'(R^2)|\partial R/\partial x_i|^2] \, dx \neq 0, \quad i = 1, 2, \cdots, N.$
\end{enumerate}

**Theorem 3.** The ground state $R(x; E)$ is orbitally stable if

\begin{equation}
\varphi(E) > 0.
\end{equation}
Here \( \varphi(E) \) is expressible as follows:

\[
\varphi(E) = \frac{d}{dE} \| R(E) \|^2 = -\frac{1}{E} \frac{d}{dE} \mathcal{H}[R(E)]
\]

\[= -\frac{1}{E} \frac{d}{dE} \int \left[ ER^2 - f(R^2) R^2 - G(R^2) \right] dx.
\]

For the special case \( f(|\phi|^2) = |\phi|^{2\sigma} \), (4.1) reduces to the subcritical condition \( u < \frac{2}{N} \).

To prove Theorem 3, we follow the outline of Sections 3 and 4. The required generalizations are contained in the following sequence of propositions.

**Proposition 4.1.** If \( x_0 \) and \( \gamma \) are such that (2.7) is minimized, then

\[
\int \left[ f(R^2) + 2 R^2 f'(R^2) \right] \frac{\partial R}{\partial x_j} u dx = 0, \quad j = 1, \ldots, N,
\]

\[
\int f(R^2) R \gamma dx = 0.
\]

These conditions reduce to (3.1) and (3.2) for the special case \( f(|\phi|^2) = |\phi|^{2\sigma} \).

In analogy with the analysis of Sections 2 and 3, relations (4.3) and (4.4) imply (2.9) provided (4.1) holds. The key ingredient is a generalization of Proposition 3.1, for which we require the following

**Proposition 4.2.** \( L_+ \) has exactly one negative eigenvalue.

Proof: For \( N = 1 \) the proof is simple. Since \( R \) is positive and symmetric, \( R' \) has a single node at \( x = 0 \). By oscillation theory for ordinary differential equations, 0 is the second eigenvalue of \( L_+ \). Therefore, there is exactly one negative eigenvalue.

For \( N \geq 2 \) we use a variational method. We first consider the special case \( f(|\phi|^2) = |\phi|^{2\sigma} \). A calculation of \( \delta^2 J[u] \|_{u=R} \), where \( J[u] \) is defined in (1.5), shows that \( L_+ + r_1 \) is a non-negative operator, where \( r_1 \) is an operator of rank one. Therefore, \( L_+ \) can have at most one negative eigenvalue. Since \( \nabla R \) is not positive, it cannot be the ground state. Hence, there is exactly one negative eigenvalue.

For general nonlinearities \( f \) we proceed by a deformation argument. Consider the one-parameter family of equations

\[
\Delta u_\tau - u_\tau + (1 - \tau) u^{2\sigma+1}_\tau + \tau f(u^2_\tau) u_\tau = 0.
\]

Let \( R_\tau \), in accordance with (B1), be the unique ground state of (4.5), and denote
by $L_+(\tau)$ the linearized operator. We have shown above that $L_+ = L_+(0)$ has exactly one negative eigenvalue and wish to conclude the same for $L_+(1)$. By (B2) the null space of $L_+(\tau)$ has dimension $N$ for any $\tau$ and is spanned by $\partial R_+ / \partial x_j$. Since $\partial R_+ / \partial x_j$ is not the ground state, $L_+(\tau)$ has at least one negative eigenvalue for any $\tau \in [0, 1]$. Now the eigenvalues of $L_+(\tau)$ vary continuously with $\tau$, so the only way a second negative eigenvalue can arise is by passage along the real line through zero. Since this would cause the multiplicity of zero, as an eigenvalue, to increase, this cannot happen.

Proposition 3.1 generalizes as follows.

**Proposition 4.3.**

\[(4.6) \quad \alpha = \inf_{(f, R(E)) = 0} (L_+ f, f) = 0 \quad \text{if} \quad \varphi(E) > 0.\]

**Proof:** Since $L_+ \nabla R = 0$ and $(f, \nabla R) = 0$ we have $\alpha \leq 0$. It can be shown (see [24]), that $\alpha$ is attained at a function $f_*$, for which

\[(4.7) \quad (L_+ - \alpha) f_* = \beta R,\]

\[(4.8) \quad \|f_*\| = 1,\]

and

\[(4.9) \quad (f_*, R) = 0.\]

It suffices to show that $\alpha \geq 0$. Otherwise, if $\beta = 0$, then $\alpha$ is an eigenvalue of $L_+$. This implies, by Proposition 4.2, that $f_*$ is the ground state. Thus $f_* > 0$ which contradicts (4.9). Now, since $\beta \neq 0$, $\alpha$ cannot be the lowest eigenvalue of $L_+$, for if it were, the equation obtained by taking the inner product of (4.7) with the ground state of $L_+$ would contradict (4.9) again.

We now consider the function

\[(4.10) \quad g(\lambda) = \left((L_+ - \lambda)^{-1} R, R\right)\]

which is well defined and smooth for $\lambda \in (\lambda_0, 0)$, where $\lambda_0$ is the lowest eigenvalue of $L_+$. For (4.9) to hold, we need $g(\alpha) = 0$. We shall show that this cannot happen for $\alpha$ negative if $\varphi(E) > 0$.

Differentiation of (4.10) gives

\[(4.11) \quad g'(\lambda) = \left\| (L_+ - \lambda)^{-1} R \right\|^2.\]

Thus, $g$ is an increasing function on $(\lambda_0, 0)$. Therefore, a sufficient condition for there to be no zero of $g(\lambda)$ on $(\lambda_0, 0)$ is $g(0) = (L_+^{-1} R, R) \leq 0$. This condition can be rewritten as (4.1)–(4.2).
**Proposition 4.4.** Let $R(\cdot, E)$ be such that (4.1) holds. Then

\begin{equation}
(4.12) \quad \inf \{ (L_+ f, f) : \|f\| = 1, (f, R(E)) = 0, \text{ and (4.3) holds} \} > 0.
\end{equation}

Proof: This follows from Proposition 4.3 and (B3).

Propositions 4.1–4.4 are the necessary tools for the generalization of all stability estimates of Section 3 through Proposition 3.3. The proof is concluded as follows. We first restrict ourselves to perturbations of the ground state satisfying (3.5), and argue by the triangle inequality as in (3.17). Here, $\psi_\lambda$ is replaced by $R(\cdot, \lambda(E))$, the ground state for which $\|R(E)\| = \|\phi_0\|$. Since there is no dilation invariance (3.15) for general nonlinearities, we obtain $R(\cdot, \lambda(E))$ via the alternative characterization

\begin{equation}
(4.13) \quad \inf_u \int \left[ |\nabla u|^2 - G(|u|^2) \right] dx, \quad u \in H^1, \|u\| = E.
\end{equation}

If $f$ satisfies (A1)–(A3) the minimum in (4.13) is attained.

5. Stability of the Solitary Traveling Wave for a Gkdv

The initial value problem for equations like Gkdv has been studied by Dushane [10], Strauss [20], and Kato [15]. Solitary waves of Gkdv may be sought of the form

\begin{equation}
(5.1) \quad w(x, t) = \psi(x - ct), \quad c > 0.
\end{equation}

Substitution of (5.1) into Gkdv, implies an ordinary differential equation for $\psi$. If we integrate this ordinary differential equation, assuming that $\psi$ and its derivatives vanish at $\pm \infty$, we have

\begin{equation}
(5.2) \quad \psi'' - c\psi + a_1(\psi) = 0,
\end{equation}

where

\begin{equation}
(5.3) \quad a_1(t) = \int_0^t a(\mu) d\mu.
\end{equation}

Integration of (5.2) yields

\begin{equation}
\frac{1}{2}(\psi')^2 - \frac{1}{2}c\psi^2 + a_2(\psi) = 0,
\end{equation}

where

\begin{equation}
(5.4) \quad a_2(t) = \int_0^t (t - \mu) a(\mu) d\mu.
\end{equation}
We now make the following assumptions on $a(t)$:

(C1) \[ t_0 \equiv \min \{ t > 0 : a_2(t) - \frac{1}{2}ct^2 = 0 \} \text{ exists,} \]

(C2) \[ ct_0 - a_1(t_0) < 0. \]

Under these assumptions there exists a unique (up to translation) ground state $\psi(\xi)$, i.e., a positive, even, monotonically decreasing solution of (5.2) that vanishes at infinity (see Berestycki and Lions [4]). In the special case $a(w) = w^p$ we have

(5.5) \[ \psi(x - ct) = \left[ \frac{1}{2}(p + 1)(p + 2) \right]^{1/2} c^{1/p} \sec h^{2/p} \left[ \frac{1}{2} pc^{1/2}(x - ct) \right]. \]

For global existence in $C([0, \infty); H^2)$ we require (see [15])

(C3) \[ \lim_{|\lambda| \to \infty} \sup |\lambda|^{-4} a(\lambda) \leq 0. \]

For stability we also require

(C4) \[ \int a(\psi(x))(\psi'(x))^2 \, dx \neq 0. \]

The solitary wave of Gkdv fits into the framework of Sections 2 and 3 for NLS. We have

**Theorem 4.** If

(5.6) \[ \varphi(c) > 0, \]

where $\varphi$ is given in (4.2), then the solitary wave $\psi(x; c)$ is orbitally stable, i.e., for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that if

(5.7) \[ \min_{x_0} \| w_0( \cdot + x_0 ) - \psi(\cdot) \|_{H^1} < \delta(\epsilon), \]

then

(5.8) \[ \min_{x_0} \| w(\cdot + x_0, t) - \psi(\cdot) \|_{H^1} < \epsilon. \]

Since the details of the proof parallel those for NLS we only give an outline. By the existence theory of Kato [15], (C3) implies the existence of a unique global solution of Gkdv of class $C([0, \infty); H^2)$. The solution $w(x, t)$ has the following
conserved integrals:

\begin{align*}
(5.9) \quad \mathcal{F}(w) &= (w(t), 1), \\
(5.10) \quad \mathcal{N}(w) &= \|w(t)\|^2, \\
(5.11) \quad \mathcal{H}(w) &= \frac{1}{2}\|Dw(t)\|^2 - (a_2(w(t)), 1).
\end{align*}

We use the conserved quantity

\begin{equation}
(5.12) \quad \mathcal{E}[w] \equiv \mathcal{H}(w) + c\mathcal{N}[w]
\end{equation}

as a Lyapunov function. Orbital stability is now stability modulo spatial translation (there is no complex phase invariance here). We therefore write

\begin{equation}
(5.13) \quad w(x + x_0, t) = \psi(x) + u(x, t),
\end{equation}

where $x_0$ is to be optimally chosen. The analogue of (2.5) is then

\begin{equation}
(5.14) \quad \mathcal{E}[w_0] - \mathcal{E}[\psi] \geq (L_+u, u) - \alpha\|u\|_{\mu}^{2+\beta}, \quad \alpha, \beta > 0,
\end{equation}

where

\begin{equation}
(5.15) \quad L_+ = -\frac{d^2}{dx^2} + c - a(\psi).
\end{equation}

The parameter $x_0$ is chosen to minimize

\begin{equation}
(5.16) \quad \|w'(\cdot + x_0, t) - \psi'(\cdot)\|^2 + c\|w(\cdot + x_0) - \psi(\cdot)\|^2.
\end{equation}

This leads to the constraint

\begin{equation}
(5.17) \quad \int a(\psi(x))\psi'(x)u(x, t) \, dx = 0.
\end{equation}

However, since $L_+$ has a negative eigenvalue, an additional constraint is needed for a lower estimate of $(L_+u, u)$. We then divide the proof into two parts: (a) first consider small perturbations of the ground state with $L^2$ norm equal to $\|\psi\|$ and (b) arbitrary small perturbations.

In analogy with NLS when $f(|\phi|^2) = |\phi|^{2\sigma}$ and $\sigma = 2/N$, in the special case $a(w) = w^p$ the stability proof breaks down when $p = 4$. No analogous instability theorem for $p \geq 4$ has been proved. However, we do have results on the role of the ground state, for the critical case $p = 4$ if, as is conjectured, the formation of singularities occurs (see [26]).
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