By analytic continuation, $f_n(x)$ is the solution of the functional equation for all x.

Also solved by J. H. Hodges, Richard Otter and Robert Weinstock, W. F. Trench, and the proposer.

Necessary and Sufficient Condition for a Polynomial

4813 [1958, 712]. *Proposed by H. D. Brunk, University of Missouri*

Given a differentiable function $f(x)$ such that to each $x \in (0, 1)$ there corresponds a positive integer $k = k(x)$ for which $f^{(n)}(x) = 0$ for all $n \geqq k$. Prove f is a polynomial.

Solution by R. P. Boas, Jr., Northwestern University. More generally, it can be proved that if for each x there is a $k = k(x)$ such that $f^{(k)}(x) = 0$, then f is a polynomial. (Corominas and Sunyer Balaguer, *Revista Mat. Hisp.-Amer.* (4) 14 (1954), 26–43; *Math. Reviews* 15, 942.) A proof is sketched in the review. Since the original paper is not readily accessible a (slightly different) proof is given here.

Let E_n be the set of points x for which $f^{(n)}(x) = 0$. Every x is in at least one E_n. By Baire's theorem there is a subinterval I in which some E_n is everywhere dense. Since $f^{(n)}$ is continuous, $f^{(n)}(x) = 0$ throughout I and f coincides in I with some polynomial. If I is not all of $(0, 1)$, repeat the reasoning with any remaining part of $(0, 1)$, and so on. We thus obtain a dense open set in each of whose component intervals f coincides with some polynomial. The complement H of this set is closed: we next show that, if not empty, it is perfect. If H is not perfect it has an isolated point, which is the common endpoint of two intervals on each of which f coincides with a polynomial. If n exceeds the degree of both polynomials, $f^{(n)}(x) = 0$ for x in both intervals, so f coincides with some polynomial in the union of the two intervals, and at their common endpoint by continuity; so the point cannot belong to H after all.

Now, since H is perfect, if it is not empty we can consider it as a complete metric space and apply Baire's theorem to it. Some E_n is then dense in some neighborhood in H, that is in the part of H that is in some interval J. In other words, there is an interval J that contains points x of H with $f^{(n)}(x) = 0$ for every such x (the same n for all x). J also contains intervals K complementary to H (since H is nowhere dense), and in each K, $f^{(m)}(x) = 0$ for some m. If $m \leqq n$, $f^{(n)}(x) = 0$ in K by differentiating. If $m > n$, we have $f^{(n)}(x) = f^{(n+1)}(x) = \cdots$ at the endpoints of K, by differentiating over H (since these endpoints are points of H). Then by integrating $f^{(m)}$ repeatedly we get $f^{(n)}(x) = 0$ throughout K. The same reasoning applies to every K, so $f^{(n)}(x) = 0$ throughout J. Thus J contains no points of H after all. This contradiction means that H was empty to begin with, so there was only one interval I and f coincides with a polynomial throughout $(0, 1)$.

Also solved by Robert Breusch and A. B. Willcox, J. M. Horváth, A. F. Kaupe, Jr., James Misho, and Henry Helson and the proposer.