A note about Fréchet-differentiability of convex functions.

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Abstract

We relate in a quantitative way the Fréchet differentiability of a convex function on a normed vector space to a measure of the continuity of its subdifferential.
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1 Introduction

It is known that if a convex function $f : X \to \mathbb{R}_+ \cup \{+\infty\}$ on a Banach space X is finite at some $x \in X$ and Fréchet-differentiable at x, then its subdifferential is lower semicontinuous at x (see for instance [2, p. 86]). It is the purpose of this note to establish a quantitative result of this kind and to prove a converse.

Let us recall some definitions. A *modulus* is a function from $\mathbb{R}_+$ into $\mathbb{R}_+ \cup \{+\infty\}$ which is null at 0 and continuous at 0. In the uses we make of modulus, we may suppose the modulus μ is nondecreasing and upper semicontinuous, replacing μ by its upper semicontinuous hull given by $\overline{\mu}(r) = \inf_{s>r} \mu(s)$ if necessary, as $\overline{\mu} \geq \mu$ and $\overline{\mu}$ is still a continuous at 0. The *Hausdorff-Pompeiu excess* of a nonempty subset E of X over another subset F of X is defined by $e(E, F) := \sup_{x \in E} d(x, F)$ (with the convention $\sup \emptyset = 0$), where $d(x, F) := \inf\{\|x - y\| : y \in F\}$ (with the convention $\inf \emptyset = +\infty$).

A multimapping F from X into X^* is said to be *lower semicontinuous* at $(x, x^*) \in X \times X^*$ on its domain D if $d(x^*, F(w)) \to 0$ as $w \to x$, $w \in D$. A modulus γ such that

$$d(x^*, F(w)) \leq \gamma(\|w - x\|) \qquad \forall w \in D \tag{1}$$

will be called a *modulus of lower semicontinuity of* F at (x, x^*). This notion coincides with the usual notion of lower semicontinuity at (x, x^*) if, and only if D is a neighborhood of x.

A multimapping F from X into X^* is said to be *lower h-continuous* at x on its domain D if $e(F(x), F(w)) \to 0$ as $w \to x, w \in D$. A modulus μ such that

$$e(F(x), F(w)) \leq \mu(\|w - x\|) \qquad \forall w \in D \tag{2}$$

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will be called a *modulus of lower h-continuity of* F at x. Clearly, it is also a modulus of lower semicontinuity of F at (x, y) for any $y \in F(x)$.

We say that a modulus δ is a *modulus of Fréchet-differentiability* of a function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ finite at $x \in X$ if there exists some $x^* \in X^*$ such that for any $w \in \mathrm{dom} f$ one has

$$|f(w) - f(x) - \langle x^*, w - x\rangle| \leq \delta(\|w - x\|)\,\|w - x\|\,. \tag{3}$$

In such a case, we say that f is Fréchet-differentiable at x on its domain, with derivative $f'(x) = x^*$. Then, the restriction of x^* to the linear subspace generated by $\mathrm{dom} f - x$ is uniquely defined. Conversely, if f is Fréchet-differentiable at x on its domain with derivative x^* then there exists some modulus δ such that (3) is satisfied. When x is an interior point of the domain of f, one recovers the classical definition.

2 The equivalences

Proposition 1 *Let f be a lower semicontinuous convex function from X into $\mathbb{R} \cup \{+\infty\}$ and let $x \in \mathrm{dom} f$ with $\partial f(x)$ nonempty and let $x^* \in X^*$. Among the following assertions one has the implications (a)⇒(b)⇒(c); if moreover x is an interior point to* $\mathrm{dom} f$, *these three assertions are equivalent:*

(a) the subdifferential ∂f of f is lower h-continuous at x on its domain, with modulus of lower hemicontinuity μ;

(b) the subdifferential ∂f of f is lower semicontinuous at (x, x^*) on its domain, with modulus of lower semicontinuity γ;

(c) f is Fréchet-differentiable at x on its domain with derivative $f'(x) = x^*$ with modulus of differentiability δ.

Moreover, assuming (a) (resp. (b), resp. (c)) one can take $\gamma = \mu$ in (b) (resp. $\delta = \overline{\gamma}$ in (c), resp. $\mu(\cdot) = \inf_{c>1} c(c-1)^{-1}\delta(c\cdot) \leq 2\delta(2\cdot)$ in (a)).

Proof. (a)⇒(b) As already observed, the assertion is obvious since for any $x^* \in \partial f(x)$, $w \in X$ one has $d(x^*, \partial f(w)) \leq e(\partial f(x), \partial f(w)) \leq \mu(\|w - x\|)$; thus (b) holds with $\gamma = \mu$.

(b)⇒(c) Let $(x, x^*) \in \partial f$ and let γ be a modulus such that (1) holds for $F := \partial f$. For any $w \in \mathrm{dom}\partial f$ such that $\gamma(r) < +\infty$ for $r := \|w - x\|$ and for any $\varepsilon > 0$ one can find $w^* \in \partial f(w)$ satisfying the inequality

$$\|w^* - x^*\| \leq d(x^*, \partial f(w)) + \varepsilon \leq \gamma(r) + \varepsilon. \tag{4}$$

Then, by definition of $\partial f(w)$, we have

$$f(w) - f(x) - \langle x^*, w - x\rangle \leq \langle w^*, w - x\rangle - \langle x^*, w - x\rangle \leq \|w^* - x^*\|\,\|w - x\|\,.$$

Since the left hand side is nonnegative, taking (4) into account and passing to the infimum over $\varepsilon > 0$, we get (3) with $\delta := \gamma$ whenever $\gamma(\|w - x\|)$ is finite; when $\gamma(\|w - x\|)$ is infinite relation (3) is obvious. Using the lower semicontinuity of f and the fact that $\mathrm{dom}\partial f$ is dense in $\mathrm{dom} f$ we get assertion (c) with $\delta := \overline{\gamma}$.

(c)⇒(a) when $D := \mathrm{dom} f$ is a neighborhood of x. Let $\rho > 0$ be such that $B(x,\rho) \subset D$. Let $x^* := f'(x)$ and let $r \in (0, \rho)$ be such that $\mu(r) < +\infty$ for μ given by $\mu(r) := \inf_{c>1} c(c-1)^{-1}\delta(cr)$. Given $\lambda > \mu(r)$ we can find $c > 1$ such that $c(c-1)^{-1}\delta(cr) < \lambda$. For any $w \in \mathrm{dom}\partial f$ such that $\|w - x\| = r$ and for any v in the closed ball $B(0, cr - r)$ with center 0 and radius $cr - r$, let us set $u := v + w - x \in B(0, cr)$ in (3). Then, for any $w^* \in \partial f(w)$ we have

$$f(w) - f(x+u) + \langle w^*, x + u - w\rangle \leq 0. \quad (5)$$
$$f(x+u) - f(x) - \langle x^*, u\rangle \leq cr\delta(cr), \quad (6)$$
$$f(x) - f(w) + \langle x^*, w - x\rangle \leq 0 \quad (7)$$

Adding the respective sides of the three preceding inequalities, we get

$$\langle w^* - x^*, v\rangle = \langle w^*, x + u - w\rangle - \langle x^*, u\rangle + \langle x^*, w - x\rangle \leq cr\delta(cr),$$

hence, taking the supremum over $v \in B(0, cr - r)$, we obtain

$$(c-1)\|w^* - x^*\| \leq c\delta(cr).$$

so that $\|w^* - x^*\| \leq \lambda$. Taking the supremum over $w^* \in \partial f(w)$ and the infimum over $\lambda \in (\mu(r), +\infty)$, we get the announced estimate on $\mathrm{dom}\partial f$.

Remark. There is a certain analogy between the estimate of μ in terms of δ and the estimates of Proposition 8.5 of [3] which states that if X is a dual Banach space, if $g : X \to \mathbb{R}_+ \cup \{+\infty\}$ is a weak* lower semicontinuous convex function having a nonempty set of minimizers S which satisfies

$$\psi(d(w, S)) \leq \inf_{u\in S} \sup_{w^*\in\partial g(w)} \langle w^*, w - u\rangle, \quad \forall w \in \mathrm{dom}\partial g\backslash S,$$

then one has

$$g(w) \geq \inf g(X) + \sup_{c\in(0,1)} c^{-1}(1-c)\psi(cd(w, S)) \quad \forall w \in X\backslash S.$$

Using duality results ([1], [5], [6]) and the fact that for any $x^* \in \partial f(x)$ the function $w \mapsto f(w) - f(x) - \langle x^*, w - x\rangle$ attains its minimum on X at x, these two results could probably be related.

Corollary 2 *Let $f : W \to \mathbb{R}$ be a convex function on some convex open subset W of a normed vector space X. If f is Fréchet differentiable at some $x \in W$ and Gâteaux differentiable on W, then its derivative is continuous at x.*

In the following corollary we use the fact that the subdifferential of a lower semicontinuous convex function f is upper semicontinuous from X endowed with the strong topology into X^* equipped with the bounded weak-star topology in the following sense: for any net $(x_i)_{i\in I}$ converging to some x in the domain of f and for any bounded net $(x_i^*)_{i\in I}$ weak* converging to some $x^* \in X^*$ one has $x^* \in \partial f(x)$ whenever $x_i^* \in \partial f(x_i)$ for each $i \in I$. We say that ∂f is *continuous at* $x \in \mathrm{dom}\partial f$ for the Mosco convergence if it is lower semicontinuous at x on its domain and upper semicontinuous from X endowed with the strong topology into X^* equipped with the bounded weak-star topology.

Corollary 3 *A lower semicontinuous convex function f from X into $\mathbb{R} \cup \{+\infty\}$ is Fréchet-differentiable at some point x of its domain iff the multimapping ∂f from X into X^* is continuous at $x \in \mathrm{dom}\partial f$ for the Mosco convergence.*

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