On the zeros of the zeta function and eigenvalue problems

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Abstract. In this paper we provide a proof of the Riemann Hypothesis by relating the non-trivial zeros of the zeta function to a certain Sturm-Liouville eigenvalue problem on the unit interval.

1 Main result

The Riemann Hypothesis (RH) is the conjecture that the non-trivial zeros of the Riemann zeta function $\zeta(s)$ all have real part equal to $\frac{1}{2}$. In [3] it was established that the function $\zeta(s)$ which is for $\Re(s) > 1$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be extended to a meromorphic function with a unique simple pole at $s = 1$, by the identification of an entire function $\xi(s)$ that satisfies the functional equation

$$\xi(s) = \xi(1 - s)$$

and that is given in terms of $\zeta(s)$ and the Gamma function $\Gamma(s)$ by

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

RH is phrased as follows in terms of $\Xi(t) = \xi(\frac{1}{2} + \mathbf{i}t)$:

$$\Xi(t) = 0 \quad \Rightarrow \quad t \in \mathbf{R}.$$

Theorem 1 *RH holds true, and the zeros $t_1, t_2, \ldots$ are such that*

$$\forall \varepsilon > 0 : \quad \sum_{n=1}^{\infty} |t_n|^{-1-\varepsilon} < \infty. \tag{1}$$

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A function similar to $\Xi(t)$ was shown in [2] to have only real zeros, by linking this function to a certain operator of Sturm-Liouville type. Inspired by this approach, we identify another Sturm-Liouville problem, the eigenfunctions of which are related to $\Xi(t)$, and deduce by drawing on classical results that the zeros t of $\Xi(t)$ are real; the eigenvalue problems are presented in Section 2, while the proof of Theorem 1 is given in Section 3.

2 A class of eigenvalue problems

Let $D = C^2([0,1],\mathbf{C})$ denote the set of C^2 functions $f : [0,1] \to \mathbf{C}$ and let p, q and $r : [0,1] \to \mathbf{C}$ be Borel functions that are such that

$$\int_0^1 \{\,|p(x)| + |q(x)| + |r(x)|\,\}\,\mathrm{d}x < \infty. \tag{2}$$

For $f \in D$ denote by f' the derivative of f and consider the following boundary problem of Sturm-Liouville type on the unit interval $(0,1)$:

$$\begin{gathered} \left(\frac{f'}{r}\right)' + (\lambda^2 p + q)\,f = 0, \\ f(0) = 1, \quad f(1) = 0, \\ f \in D, \quad \lambda \in \mathbf{C}. \end{gathered} \tag{3}$$

The following result which concerns the eigenfunctions $u(\,\cdot\,,\lambda)$ of (3) and the corresponding eigenvalues λ^2 is well-known (the statement is taken from [1, Theorem 8.3.1]):

Theorem 2 *Suppose that p,q and r are (a) nonnegative, (b) satisfy* (2)*, and are such that (c) for any x, $0 < x < 1$,*

$$\int_0^x p(z)\,\mathrm{d}z > 0, \quad \int_x^1 p(z)\,\mathrm{d}z > 0, \quad \int_0^1 r(z)\,\mathrm{d}z > 0$$

and (d) if for some x_1, x_2, $0 \le x_1 < x_2 \le 1$ we have

$$\int_{x_1}^{x_2} p(z)\,\mathrm{d}z = 0,$$

then

$$\int_{x_1}^{x_2} |q(z)|\,\mathrm{d}z = 0.$$

Then the boundary problem (3) *has at most countably many eigenvalues* $\lambda_1^2,\ \lambda_2^2, \ldots$, *all of which are real, and which are such that*

$$\forall \varepsilon > 0 : \qquad \sum_{\lambda_n \neq 0} |\lambda_n|^{-1-\varepsilon} < \infty. \tag{4}$$

The eigenfunctions $u(x, \lambda_n)$ *are orthogonal in the sense that*

$$\int_0^1 p(x)\, u(x, \lambda_n)\, u(x, \lambda_m)\, \mathrm{d}x = 0, \qquad n \neq m.$$

We next present a particular family of solutions to (3). For any $\mu \in \mathbf{C}$ and non-negative Borel-measurable function $b : \mathbf{R}_+ \to \mathbf{R}_+$ that satisfies

$$(a) \quad \int_0^\infty b(y)\, \mathrm{d}y \;=\; 1; \quad (b) \quad \forall t \in \mathbf{R}_+ : \qquad \int_0^\infty \exp\{ty\}\, b(y)\, \mathrm{d}y \;\in\; \mathbf{R}_+, \tag{5}$$

define the function $v_b(\,\cdot\,,\,\mu\,) : [0, 1] \to \mathbf{C}$ by

$$v_b(x, \mu) \;=\; \int_0^\infty \cos\left(\mu \left(\frac{\exp\{x\} - 1}{\exp\{1\} - 1} \right) y \right) b(y)\, \mathrm{d}y. \tag{6}$$

Lemma 3 *For* $\mu \in C$ *that is such that*

$$v_b(\,1\,,\,\mu\,) \;=\; 0, \tag{7}$$

$v_b(\,\cdot\,,\,\mu\,)$ *is an eigenfunction of* (3) *with eigenvalue* μ^2 *which is real.*

Proof. Let p, q and r be given by

$$\begin{aligned} r(x) &= \exp\{x\}, \\ p(x) &= \exp\{-x\} \left(\frac{\exp\{x\}}{\exp\{1\} - 1} \right)^2, \\ q(x) &= 0. \end{aligned}$$

As integration and differentiation may be interchanged under the integrability condition (5), it is straightforward to check that $v = v_b(\cdot, \mu)$ is such that $v(0) = 1$ and $v(1) = 0$ and for any $0 < x < 1$

$$\begin{aligned} v''(x) &= -\,\mu^2 \left(\frac{\exp\{x\}}{\exp\{1\} - 1} \right)^2 v(x) + v'(x) \\ &= v(x)\,[\,-\mu^2\, r(x)\, p(x) - r(x)\, q(x)\,] \;+\; v'(x) \left[\frac{r'(x)}{r(x)} \right], \end{aligned}$$

Hence $v_b(\cdot, \mu)$ is an eigenfunction of (3) with eigenvalue μ^2 which is real by Theorem 2. QED

3 Proof of Theorem 1

As noted in [3], Ξ admits the following expression as Fourier cosine integral:

$$\begin{aligned} \Xi(t) &= 4\int_1^\infty \frac{\mathrm{d}(x^{\frac{3}{2}}\psi'(x))}{\mathrm{d}x}\, x^{-\frac{1}{4}} \cos\left(\frac{t}{2}\log x\right)\mathrm{d}x, \\ \psi(x) &= \sum_{n=1}^{\infty} \exp\{-n^2\pi x\}. \end{aligned} \tag{8}$$

An exponential change-of-variables in (8) yields the following well-known alternative expression for $\Xi(t)$:

$$\begin{aligned} \Xi(t) &= 2\int_0^\infty \cos(tx)\,\Phi(x)\,\mathrm{d}x, \\ \Phi(x) &= 2\pi\,\exp\left\{\frac{5x}{2}\right\}\sum_{n=1}^{\infty}\Big(2\pi\exp\{2x\}n^2-3\Big)\,n^2\exp\{-n^2\pi\exp\{2x\}\}. \end{aligned}$$

As $\Phi(x)$ has the asymptotic decay (where $f(x) \sim g(x)$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$)

$$\Phi(x) \sim 4\pi^2\,\exp\left\{\frac{9x}{2} - \pi\exp\{2x\}\right\}, \quad x \longrightarrow \infty,$$

it satisfies the exponential integrability condition in (5) (with $b(x)$ replaced by $\Phi(x)$). Also, $\Phi(x)$ is positive for any $x \in \mathbf{R}_+$. In terms of the function $\Psi(x)$ given by $2\Phi(x)/\Xi(0)$ we have from Lemma 3 that

$$\{\, t^2 \in \mathbf{C} :\ v_\Psi(1,t) = 0 \,\} \subset \mathbf{R}. \tag{9}$$

The observation

$$\forall t \in \mathbf{C}: \qquad \Xi(t) = \Xi(0)\,v_\Psi(1,t)$$

in conjunction with (9) and the fact that $\Xi(t) > 0$ for all t with $\Re(t) = 0$ completes the proof of RH. Eq. (1) follows by combining Theorem 2 and Lemma 3. QED

References

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