Gaspard Monge and the Monge Point of the Tetrahedron

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Monge and his contributions to mathematics

Gaspard Monge (1746–1818) was a man of extraordinary talent. Despite humble origins, he founded one new branch of mathematics, made major early contributions to a second, and became a close friend of Napoleon Bonaparte (1769–1821).

Monge was born on May 9 or 10, 1746, in Beaune, France [15, p. 9]. His father, a peddler and later a storekeeper, valued education and saw to the education of his three sons. By age 14, Monge made his exceptional ability evident by independently constructing a fire engine. At the Collège de la Trinité in Lyon, he so impressed his teachers that they invited him to teach physics at age 16 or 17. In the summer of 1764, at age 18, Monge returned to Beaune. There, having devised his own plans of observation and constructed his own surveying instruments, he created with remarkable skill and care a large-scale map of his hometown. (The original is still at the Beaune library.) Word of this map reached the prestigious École Royale du Génie de Mézières (Mézières Royal School of Engineering), and a high-ranking officer there offered Monge a position as a draftsman. Although Monge was unaware that he could not become a student officer because he was a commoner by birth, his decision to accept the offer turned out to be a good one.

Monge was politically active, and held several government posts. From 1783 to 1789 he was an examiner of naval cadets, and from August or September 1792 until April 1793 he was Minister of the Navy, a position made difficult by the troubles and failures of the French navy, which made Monge a target of criticism. Shortly after resigning as Minister of the Navy, Monge began supervising armaments factories and writing instruction manuals for the workers. Monge supported the French Revolution, but in the turmoil that developed, many people had unjust accusations leveled against them and were executed. Monge himself was sometimes in danger, and at one point, after being denounced by the porter at his lodgings, he left Paris. In 1796, Napoleon wrote to Monge to offer him his friendship and a position. The two had met when Monge was Minister of the Navy and Napoleon, then a little-known artillery officer, had been impressed by how Monge had treated him. Monge was sent to Italy to obtain artworks for France, and during 1798 and 1799 he accompanied Napoleon on his Egyptian campaign. After establishing the Consulate in 1799, Napoleon named Monge a senator for life.

Monge’s strong sense of justice and equality, his honesty, and his kindness were evident throughout his political career. As an examiner of naval cadets Monge rejected outright unqualified sons of aristocrats. He spoke frankly with Napoleon, and may have exerted a moderating influence on him [2, pp. 190, 196, 204]. When Napoleon returned from exile in Elba, for instance, Monge successfully counseled against taking excessive vengeance. Sadly for Monge, he was stripped of his honors after Napoleon’s final fall from power in 1815. He died in Paris on July 28, 1818.

Monge held several teaching positions, beginning with his teaching duties in physics at age 16 or 17. In 1769, at age 22, Monge became a mathematics professor at the École Royale du Génie de Mézières; by 1772 he was teaching physics as
well. After his election to the Académie des Sciences in 1780, Monge began to spend long periods in Paris. Doing so allowed him to teach hydrodynamics at the Louvre (this position had been created by Anne Robert Turgot (1727–1781), a statesman and economist [15, p. 23]), but it also forced him to resign his post at Mézières in 1784. During 1794–1795 Monge taught at the École Normale de l’an III, and starting in 1795, and again after returning from Egypt, he taught at the École Polytechnique. Declining health finally forced him to give up teaching in 1809.

Monge was an influential educator for several reasons. One is his close association with the École Polytechnique. Monge successfully argued for a single engineering school rather than several specialized schools, helped to develop the curriculum, talked with professors, advised administrators, and supervised the opening of the school. In 1797, Monge was appointed Director. With Monge, Laplace, and Lagrange among its first faculty, the École Polytechnique was influential from the beginning, and it remains France’s best-known technical school.

A second reason for Monge’s influence is that he was an exceptional and inspiring teacher, well liked and respected by his students. Monge was, according to Boyer, “perhaps the most influential mathematics teacher since the days of Euclid.” [4, p. 468] Among his students who made mathematical contributions of their own were Lazare Carnot (1753–1823), Charles Brianchon (1785–1864), Jean Victor Poncelet (1788–1867), Charles Dupin (1784–1873), J. B. Meusnier (1754–1793), E. L. Malus (1775–1812), and O. Rodrigues (1794–1851).

Monge’s major mathematical contributions were in the areas of descriptive geometry, differential geometry, and analytic geometry. His work in descriptive geometry began in his early days at Mézières, when he was assigned to produce a plan for a fortress that would hide and protect its defenders from enemy attack. At the time such problems were solved by long computations, but Monge used a geometrical method to solve the problem so quickly that his superior officer at first refused to believe Monge’s results. (For details on the problem and on Monge’s solution, see Taton’s account [15, pp. 12–14].) When the solution was found to be correct, it was made a military secret and Monge was assigned to teach the method. (He was not allowed to teach the method publicly until 1794, at the École Normale de l’an III.)

In 1802, Monge and Jean-Nicolas-Pierre Hachette (1769–1834) published Application d’algèbre à la géométrie, in the Journal de l’École Polytechnique. (This work appeared again in 1805 and 1807.) It summarized Monge’s lectures in solid analytic geometry. Boyer remarks: “The notation, phraseology, and methods are virtually the same as those to be found in any textbook of today. The definitive form of analytic geometry finally had been achieved, more than a century and a half after Descartes and Fermat had laid the foundations.” [3, p. 220]

Monge published two papers [11, 12] entitled “Sur la pyramide triangulaire,” in 1809 and 1811. In the remainder of the present paper, we focus mainly on the content of these two papers. A detailed account of all of Monge’s work on the tetrahedron is given by Taton [15, pp. 241–246].

The Monge point of the tetrahedron

Two edges of a tetrahedron are called opposite edges if they have no common vertex. A tetrahedron $ABCD$ has three pairs of opposite edges; one pair is $AB$ and $CD$. With this convention, we can define six Monge planes, one for each edge:

**Definition.** A Monge plane is perpendicular to one edge of a tetrahedron and contains the midpoint of the opposite edge.
The following theorem guarantees the existence of a Monge point, where the Monge planes meet.

**Theorem 1.** The six Monge planes of a tetrahedron are concurrent.

**Example:** Figure 1 shows the tetrahedron with vertices at $O(0, 0, 0)$, $A(4, 0, 0)$, $B(0, 4, 0)$, and $C(0, 0, 4)$. The six Monge planes are perpendicular to any given face of the tetrahedron and therefore intersect in a line perpendicular to the face. For example, the Monge planes $x = 0$, $y = 0$, and $x = y$ are perpendicular to face $AOB$ and intersect in the $z$-axis. The three Monge planes perpendicular to face $ABC$ intersect in the line $x = y = z$.

The following proof differs only in details from the one given by Monge [12, pp. 263–265]. A similar proof is given by Thompson [16].

**Proof.** In tetrahedron $ABCD$, construct a triangle $B'C'D'$ by connecting the midpoints of the three edges issuing from vertex $A$ (as in Figure 2). Each of the three Monge planes perpendicular to an edge of face $BCD$ cuts the triangle $B'C'D'$ in one of its altitudes, for each contains a vertex of this triangle and cuts its opposite side perpendicularly. Therefore, these Monge planes intersect in the perpendicular from the orthocenter $H'$ of triangle $B'C'D'$ to face $BCD$. Let us call this line the Monge normal of face $BCD$. By the same reasoning, each face of the tetrahedron has a Monge normal associated with it.

The Monge normals of faces $ABC$ and $ACD$ intersect at a point $M$, for they are not parallel and they both lie in the Monge plane perpendicular to edge $AC$. We can now be certain that $M$ lies on every Monge plane except possibly the one perpendicular to edge $BD$. Since $M$ lies on the Monge planes perpendicular to edges $AB$ and $AD$, it lies on their line of intersection. But their line of intersection is the Monge normal of face...
Figure 2  A tetrahedron $ABCD$ with a midpoint triangle $B'C'D'$ parallel to face $BCD$. Face $BCD$ lies in the plane of the paper and the view is from directly above the tetrahedron. Intersections involving the three Monge planes perpendicular to the edges of face $BCD$ are marked as follows: thin solid lines indicate the intersections of the Monge planes with that portion of the surface of the tetrahedron lying between face $BCD$ and triangle $B'C'D'$; dashed lines indicate the intersections with $B'C'D'$.

$ABD$, which lies in the Monge plane perpendicular to $BD$. Hence, $M$ lies on all six Monge planes, and the theorem is established.

The Monge point, the circumcenter, and the centroid

Monge actually discovered and proved more than Theorem 1. Before stating his more general result, we will describe relationships among the bimedians and the centroid; relate the centroid to the center of mass and the center of gravity; define the circumcenter; and give an example.

DEFINITION. A bimedian is a line segment connecting the midpoints of opposite edges of a tetrahedron.

LEMMA. The three bimedians of a tetrahedron bisect each other.

We give analytic and synthetic proofs. The first proof is very simple; Monge himself gave the second [11, p. 2].

Analytic proof. Write the four vertices as $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$, and $D(d_1, d_2, d_3)$. Straightforward uses of the midpoint formula (three for each bimedian) show that the midpoint of each bimedian is the point $((a_1 + b_1 + c_1 + d_1)/4, (a_2 + b_2 + c_2 + d_2)/4, (a_3 + b_3 + c_3 + d_3)/4)$.  

$\blacksquare$
Figure 3 Tetrahedron $ABCD$ (thick solid lines), its circumscribed parallelepiped (thin solid lines), and its three bimedians (dashed lines). The bimedians are the axes of the parallelepiped, so they bisect each other at the center of the parallelepiped.

**Synthetic proof.** Through each edge of the tetrahedron, construct a plane that is parallel to the opposite edge. (To construct the desired plane through, say, edge $AB$, construct a line through $AB$ that is parallel to edge $CD$.) This process produces a parallelepiped that circumscribes the tetrahedron. (See Figure 3.) Each edge of the tetrahedron is a diagonal of a face of the parallelepiped; opposite edges of the tetrahedron are diagonals of opposite faces of the parallelepiped. Hence, the three bimedians of the tetrahedron are the axes of the parallelepiped (the segments connecting the centers of opposite faces). Therefore, the bimedians bisect each other at the center of the parallelepiped, and the proof is complete.

Before stating the next theorem, we note that the centroid is sometimes referred to as the center of mass or the center of gravity. Monge himself [11, 12] uses the term “center of gravity.” The centroid of a solid coincides with its center of mass if the solid has uniform density. (The centroid of a solid is also known as its center of volume.) The center of gravity of a solid coincides with its center of mass if the solid is subjected to a uniform gravitational field. Hence, for the case of an ideal tetrahedron that is both of uniform density and subjected to a uniform gravitational field, the center of mass, center of gravity, and centroid all coincide.

Polya discusses the center of gravity (centroid) of the tetrahedron in an intuitive manner [14, pp. 38–45].

**Theorem 2.** The centroid of a tetrahedron lies at the common midpoint of its three bimedians.
This theorem plays a crucial role in what follows. Monge [11] gave two complete proofs of it, and also stated a previously-known theorem from which he said the result could be easily derived. (The theorem asserts that \( d_G = (d_A + d_B + d_C + d_D)/4 \), where the variables are the signed distances of the centroid \( G \) and the vertices from an arbitrary plane, the distances being positive for points on one side of the plane and negative for points on the other.) Following are some key ideas from Monge’s first proof.

Let us consider a tetrahedron as an infinite collection of line segments, all parallel to one edge of the tetrahedron. A median plane of a tetrahedron contains one edge and the midpoint of the opposite edge. If a median plane bisects the edge that the line segments are parallel to, it will bisect all of the line segments that constitute the tetrahedron. (See Figure 4.) It follows that the centers of mass of all of the line segments—and therefore also the center of mass of the entire tetrahedron—lie on the median plane. Since we may consider line segments parallel to any edge of the tetrahedron, we conclude that the centroid lies on all six median planes. But median planes that contain opposite edges intersect in a bimedian. Hence, the centroid of the tetrahedron lies on each of the three bimedians. Since, by the Lemma, the bimedians bisect each other, the centroid is their common midpoint.

Figure 4  An arbitrary segment \( s \) that is parallel to edge \( AD \) and that has its endpoints in faces \( ABC \) and \( BCD \). Triangle \( ADX \) is the intersection of the tetrahedron with the plane of edge \( AD \) and segment \( s \). Median plane \( BCM_AD \) bisects segment \( s \) because it contains \( XM_{AD} \), which is a median of triangle \( ADX \) and bisects \( s \).

The centroid of a tetrahedron can be defined as the intersection of its bimedians or as the intersection of its four medians (the line segments from a vertex to the centroid of the opposite face). Monge related these two definitions [11, p. 4]. If two median planes contain opposite edges of a tetrahedron, they will intersect in a bimedian; if median planes contain edges issuing from the same vertex of a tetrahedron, they will intersect in a median. To see this consider the three median planes \( ABM_{CD}, ACM_{BD}, \) and \( ADM_{BC} \). Since \( BM_{CD}, CM_{BD}, \) and \( DM_{BC} \) are the medians of face \( BCD \), each of these median planes contains the centroid \( G_{BCD} \) of face \( BCD \). They therefore intersect in the median \( AG_{BCD} \) of the tetrahedron.
The *circumcenter* of a tetrahedron is the center of the sphere defined by the four vertices of the tetrahedron. The circumcenter is equidistant from the four vertices and lies at the intersection of the perpendicular bisectors of the six edges of the tetrahedron.

For the tetrahedron in Figure 1 the centroid is $(1, 1, 1)$ (by Theorem 2) and the circumcenter is $(2, 2, 2)$. Since the Monge point is at the origin, it is seen to be symmetric to the circumcenter with respect to the centroid. A similar property holds for every tetrahedron:

**MONGE’S THEOREM.** The six Monge planes of a tetrahedron are concurrent at the reflection of the circumcenter in the centroid.

An analytic proof is straightforward, provided the origin is placed at the circumcenter of the tetrahedron. The coordinates of the reflection of the circumcenter in the centroid, $M$, are then double the coordinates of the centroid (which were given in the analytic proof of the Lemma). One can show that $M$ satisfies the equation of any of the Monge planes by substituting the coordinates of $M$ into the equation, simplifying, and using the fact that the vertices of the tetrahedron are equidistant from the origin. Eves gives a complete analytic proof along these lines [7, p. 149]. Forder gives a somewhat different analytic proof [10, p. 471].

The following synthetic proof appears in Altshiller-Court’s book [1, p. 76] (an excellent source of information on the geometry of the tetrahedron).

**Proof.** Consider the perpendicular bisector of any edge of a tetrahedron and the Monge plane perpendicular to the same edge. These two planes are parallel. A bimedian crosses between them, for the perpendicular bisector contains the midpoint of one edge and the Monge plane contains the midpoint of the opposite edge. The centroid of the tetrahedron lies midway between the planes, for the centroid is the midpoint of the bimedian (Theorem 2). Hence, the reflection in the centroid of any point in one plane lies in the other plane. In particular, since the circumcenter lies in the perpendicular bisector, its reflection in the centroid is a point $M$ lying in the Monge plane. By this reasoning, $M$ lies on all six Monge planes and is the Monge point.

![Figure 5](image)

**Figure 5** The plane of the circumcenter $O$, the centroid $G$, the Monge point $M$, and the midpoints $M_{AB}$ and $M_{CD}$ of edges $AB$ and $CD$ of tetrahedron $ABCD$. The perpendicular bisector of edge $AB$ intersects the plane of the figure in $M_{AB}O$; the Monge plane perpendicular to $AB$ intersects it at $M_{CD}M$.

Figure 5 depicts the plane of the circumcenter, the centroid, the Monge point, and one of the three bimedians.
Except for a brief introduction and a final short paragraph, Monge devotes “Sur la pyramide triangulaire” [12] entirely to a long two-part proof of this theorem. He begins the final paragraph with the assertion that the entire proof can be greatly simplified, but he gives us only key points of the simplified proof.

The first part of the long proof, which establishes the existence of the Monge point, we gave as our proof of Theorem 1. The second part establishes that the Monge point is the reflection of the circumcenter in the centroid, and it is not very elegant. We omit this part, but remark that it involves the construction of three new vertices related to a tetrahedron \(T\). In the final paragraph Monge extends the construction from the previous proof by adding a fourth new vertex to form a conjugate or twin tetrahedron \(T'\). He tells us that the twin tetrahedra have a common centroid, that their circumcenters are symmetric with respect to the centroid, and that the Monge point of \(T\) is the circumcenter of \(T'\). A citation to a paper in which Monge defines twin tetrahedra is given.

Why would Monge have given a long proof when he had a much shorter and more elegant proof, and why would he have given the long proof in detail and simply outlined the short one? Perhaps he wanted to stimulate the reader’s interest by setting up the simplification, to entice him into trying to fill in the details. Here we may see a hint of Monge’s gift for teaching.

Using the information Monge gives us [12, p. 266] and a complete proof using twin tetrahedra [1, p. 76], we speculate on how Monge might have filled in the details. In Figure 3 construct the twin \(T'\) of tetrahedron \(T = ABCD\) by connecting the four vertices of the parallelepiped that do not belong to \(T\). Label the vertices of \(T'\) so that the diagonals of the parallelepiped are \(AA', BB', CC',\) and \(DD'\). (See Figure 6.) Now, consider the Monge plane perpendicular to \(AB\) and containing \(M_{CD}\). This Monge plane is perpendicular to \(A'B'\) because \(AB\) and \(A'B'\) are parallel. It contains \(M_{A'B'}\) because \(M_{CD}\) and \(M_{A'B'}\) coincide, both being midpoints of diagonals of the same parallelogram.

Hence, the Monge plane perpendicular to edge \(AB\) of \(T\) is the perpendicular bisector of edge \(A'B'\) of \(T'\). Applying the same reasoning to the other Monge planes of \(T\), we establish the key idea of the proof: The six Monge planes of \(T\) are the perpendicular bisectors of \(T'\). Since the perpendicular bisectors of \(T'\) intersect at the circumcenter \(O'\) of \(T'\), we have \(M = O'\), and the existence of the Monge point is established. Now \(T\) and \(T'\) are clearly symmetric with respect to the center of the parallelepiped. Therefore, by Figure 3 and Theorem 2, \(T\) and \(T'\) are symmetric with respect to the centroid \(G\) of \(T\), whence the circumcenters \(O\) and \(O'\) are reflections in \(G\). Substituting \(M\) for \(O'\) establishes the symmetry of \(M\) and \(O\) with respect to \(G\) and completes the proof.

In the proof just given we guessed that Monge would have used the circumscribing parallelepiped. This seems plausible because Monge defined twin tetrahedra in terms of the circumscribing parallelepiped in another paper [15, p. 242], and a figure such as Figure 6 may help one see the theorem. However, use of the circumscribing parallelepiped can be avoided in the above paragraph with the following changes. (1) Define \(T'\) as the reflection of \(T\) in \(G\). (2) Replace the sentence, “It [the Monge plane perpendicular to \(AB\) and containing \(M_{CD}\)] contains \(M_{A'B'}\) because \(M_{CD}\) and \(M_{A'B'}\) coincide, both being midpoints of diagonals of the same parallelogram” with, “It contains \(M_{A'B'}\) because \(M_{CD}\) and \(M_{A'B'}\) coincide, both being the reflection of \(M_{AB}\) in \(G\), \(M_{A'B'}\) because \(T\) and \(T'\) are reflections in \(G\) and \(M_{CD}\) because of Theorem 2.”

Notice that in each of the three synthetic proofs of Monge’s theorem we have discussed (single tetrahedron and twin tetrahedra with and without the circumscribing parallelepiped) Theorem 2 plays an important role.

For a different approach to proving Monge’s theorem see Thompson’s proof [16].
The rectangular tetrahedron: a special case

Monge’s theorem invites comparison between the triangle and the tetrahedron; Monge stresses this analogy [12]. In a plane triangle, the circumcenter, the centroid, and the orthocenter lie on a line called the Euler line. The centroid lies between the other two points, twice as far from the orthocenter as from the circumcenter. (Two elegant proofs of this theorem are given in books by Dörrie [6, p. 141] and Eves [9, p. 109].) By analogy, the line containing the circumcenter, the centroid, and the Monge point of a tetrahedron is called the Euler line of the tetrahedron.

This analogy raises natural questions: Must a tetrahedron have an orthocenter, that is, a point at which its four altitudes meet? If an orthocenter exits, how is it related to the Monge point? The following definition and theorem provide answers.

**Definition.** A rectangular, or orthocentric, tetrahedron is one in which opposite edges have perpendicular directions.

**Theorem 3.** A tetrahedron has an orthocenter if and only if it is rectangular. In a rectangular tetrahedron, the orthocenter is the Monge point.

**Proof.** Suppose that tetrahedron $ABCD$ has an orthocenter $H_T$. Since plane $ABH_T$ contains the altitudes to faces $BCD$ and $ACD$, it is perpendicular to both of these faces.
and to their intersection at edge $CD$. Hence, edges $AB$ and $CD$ have perpendicular directions. By the same reasoning, edges $AC$ and $BD$ and $AD$ and $BC$ have perpendicular directions and tetrahedron $ABCD$ is rectangular.

Assume that the tetrahedron is rectangular. Since opposite edges have perpendicular directions, each Monge plane must contain not only a midpoint of an edge but an entire edge of the tetrahedron. Hence, if three Monge planes are perpendicular to the edges of any one face of the tetrahedron, they contain the opposite vertex, and the Monge normal and the altitude associated with the face coincide. It now follows that the four altitudes intersect at the Monge point. This completes the proof.

Altshiller-Court [1, p. 71] gives a proof of the concurrency of the four altitudes that does not assume knowledge of the Monge point. Figure 1 provides an example of a rectangular tetrahedron. The four altitudes do indeed coincide with the Monge normals and meet in the Monge point $O$.

Mannheim's theorem

A second set of planes, named for A. Mannheim (1831–1906), also intersect at the Monge point.

DEFINITION. A Mannheim plane contains the altitude to and the orthocenter of a face of a tetrahedron.

MANNHEIM'S THEOREM. The four Mannheim planes of a tetrahedron intersect at the Monge point.

Proof. In Figure 2, let the orthocenters of face $BCD$ and triangle $B'C'D'$ be $H$ and $H'$, respectively. Note that triangle $B'C'D'$ is a dilation of face $BCD$ with center $A$ and ratio $1/2$; thus, $H'$ is the midpoint of $AH$. Consider the Mannheim plane that contains $H$ and the altitude from vertex $A$, that is, the Mannheim plane that is perpendicular to face $BCD$. The line $AH$ lies in this plane, whence $H'$ lies in it, whence the Monge normal of face $BCD$ lies in it (by our proof of Theorem 1), whence the Monge point lies in it. By the same reasoning, the Monge point lies in the other three Mannheim planes, and the proof is complete.

Two additional proofs of this theorem appear in Altshiller-Court's book [1, p. 78].

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REFERENCES

Math Bite: A Novel Proof of the Infinitude of Primes, Revisited

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We rephrase a proof of the infinitude of primes by Fürstenberg [1]. The original used arithmetic progressions as the basis for a topology on the integers. Our approach avoids the language of topology.

Recall: For $A$ a subset of the integers, the characteristic function of $A$ has value 1 for $x$ in $A$ and 0 otherwise. A periodic set of integers is one whose characteristic function is periodic.

Observe that if $S$ and $T$ are periodic sets with periods $S$ and $T$ respectively, then $S \cup T$ is periodic with a period dividing $\text{lcm}(s,t)$ and that this is easily extended to all finite unions. And observe that if $S$ is periodic, then the complement of $S$ is periodic.

THEOREM. There are infinitely many prime integers.

Proof. For each prime $p$, let $S_p = \{n \cdot p : n \in \mathbb{Z}\}$. Define $S$ to be the union of all the sets $S_p$, each of which is periodic. If this union is taken over a finite set, then $S$ is periodic and then so is its complement. But the complement of $S$ is $\{-1,1\}$, which being finite is not periodic. Hence the number of primes is infinite.

As a pedagogic coda, we note that filling in the details of our proof would be a nice exercise in a course serving as a bridge to upper division mathematics.

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