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Full Rank Factorization of Matrices

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1. Introduction

There are various useful ways to write a matrix as the product of two or three other matrices that have special properties. For example, today's linear algebra texts relate Gaussian elimination to the LU factorization and the Gram–Schmidt process to the QR factorization. In this paper, we consider a factorization based on the rank of a matrix. Our purpose is to provide an integrated theoretical development of and setting for understanding a number of topics in linear algebra, such as the Moore–Penrose generalized inverse and the Singular Value Decomposition. We make no claim to a practical tool for numerical computation—the rank of a very large matrix may be difficult to determine. However, we will describe two applications; one to the explicit computation of orthogonal projections, and the other to finding explicit matrices that diagonalize a given matrix.

2. Rank

Let $\mathbb{C}$ denote the field of complex numbers and $\mathbb{C}^{m\times n}$ the collection of m-by-n matrices with entries from $\mathbb{C}$. If $A\in\mathbb{C}^{m\times n}$, let A^* denote the conjugate transpose (sometimes called the Hermitian adjoint) of A; A^* is formed by taking the complex conjugate of each entry in A and then transposing the resulting matrix.

A very important (but not always easily discoverable) nonnegative integer is associated with each matrix A in $\mathbb{C}^{m\times n}$. The rows of A can be viewed as vectors in $\mathbb{C}^n$, and the columns as vectors in $\mathbb{C}^m$. The rows span a subspace called the *row space* of A; the dimension of the row space is called the *row rank* of A. The *column rank* of A is the dimension of the subspace of $\mathbb{C}^m$ spanned by the columns. Remarkably, the row rank and the column rank are always the same, so we may unambiguously refer to the *rank* of A. Let $r(A)$ denote the rank of A and $\mathbb{C}_r^{m\times n}$ the collection of matrices of rank r in $\mathbb{C}^{m\times n}$. We say that a matrix A in $\mathbb{C}^{m\times n}$ has *full row rank* if $r(A)=m$ and *full column rank* if $r(A)=n$. The following are some basic facts about rank that we will find useful (see **[11]**):

- for $A\in\mathbb{C}^{m\times n}$, $r(A)\le\min(m,n)$;
- for $A,B\in\mathbb{C}^{m\times n}$, $r(A+B)\le r(A)+r(B)$;
- for $A\in\mathbb{C}^{m\times n}$, $r(A)=r(A^*)=r(A^*A)$;
- for $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{n\times p}$, $r(AB)\le\min(r(A),r(B))$;
- if B and C are invertible, then $r(AB)=r(A)=r(CA)$.

3. Full rank factorizations

Let $A\in\mathbb{C}_r^{m\times n}$, with $r>0$. If we can find F in $\mathbb{C}_r^{m\times r}$ and G in $\mathbb{C}_r^{r\times n}$ such that $A=FG$, then we say that we have a *full rank factorization* of A. It is not difficult to see that every matrix A in $\mathbb{C}_r^{m\times n}$ with $r>0$ has such a factorization. One approach is to choose for F any matrix whose columns form a basis for the column space of A. Then, since each column of A is uniquely expressible as a linear combination of the columns of F, the coefficients in the linear combinations determine a unique matrix G in $\mathbb{C}^{r\times n}$ with $A = FG$. Moreover, $r = r(A) = r(FG) \leq r(G) \leq r$ so that G is in $\mathbb{C}_r^{r\times n}$.

Another approach, which will lead us to an algorithm, is to apply elementary matrices on the left of A (that is, elementary row operations) to produce the unique row reduced echelon form of A, RREF(A). In other words, we compute an invertible matrix R in $\mathbb{C}^{m\times m}$ with

$$RA = \begin{bmatrix} G_{r\times n} \\ \cdots\cdots\cdots \\ \mathbb{O}_{(m-r)\times n} \end{bmatrix},$$

where $r = r(A) = r(G)$ and $\mathbb{O}_{(m-r)\times n}$ is the matrix, consisting entirely of zeros, of $m - r$ rows and n columns. Then

$$A = R^{-1}\begin{bmatrix} G \\ \cdots \\ \mathbb{G} \end{bmatrix}.$$

Let R_1 consist of the first r columns of R and R_2 the remaining columns. Then R_1 is m-by-r and R_2 is m-by-$(m-r)$, and

$$A = \begin{bmatrix} R_1 \vdots R_2 \end{bmatrix}\begin{bmatrix} G \\ \cdots \\ \mathbb{O} \end{bmatrix} = R_1G + R_2\mathbb{O} = R_1G.$$

Now take F to be R_1. Since R^{-1} is invertible, its columns are linearly independent so F has r independent columns, and hence has full column rank. This leads us to an algorithm for computing a full rank factorization of a matrix A in $\mathbb{C}_r^{m\times n}$:

Step 1. Use elementary row operations to reduce A to row reduced echelon form RREF(A).

Step 2. Construct a matrix F from the columns of A that correspond to the columns with the leading ones in RREF(A), placing them in F in the same order as they appear in A.

Step 3. Construct a matrix G from the nonzero rows of RREF(A), placing them in G in the same order as they appear in RREF(A). Then $A = FG$ is a full rank factorization of A.

Example. If $A = \begin{bmatrix} 3 & 6 & 13 \\ 2 & 4 & 9 \\ 1 & 2 & 3 \end{bmatrix}$, then $\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So, with $G = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 3 & 13 \\ 2 & 9 \\ 1 & 3 \end{bmatrix}$, we have $A = FG$, a full rank factorization.

Full rank factorizations not only exist, but abound. Indeed, if $A = FG$ is any full rank factorization of A in $\mathbb{C}_r^{m\times n}$ and R is any invertible matrix in $\mathbb{C}^{r\times r}$, then $A = FG = FRR^{-1}G = (FR)(R^{-1}G)$ is another full rank factorization of A.

4. Pseudoinverses from full rank factorizations

The concept of a *pseudoinverse* (now beginning to appear in undergraduate linear algebra texts) has its roots in the central problem of linear algebra: solving systems of linear equations. Let's consider such a system

$$Ax = b,$$

where the coefficient matrix A is m-by-n and has rank r. If $n=m=r$, then $x=A^{-1}b$ is the unique solution. But what if A is not square, or, even if A is square, if A^{-1} does not exist? Note that, for any matrix A, if we can produce another matrix B such that $ABA=A$, then Bb will be one solution to $Ax=b$ if a solution exists. To see this suppose $Ax=b$. Then $BAx=Bb$ so $b=Ax=ABAx=A(Bb)$. Surely, if A is square and invertible then $B=A^{-1}$ has the property $ABA=AA^{-1}A=A$. The matrix B is called a *generalized inverse* of A and is not necessarily unique. The first published work on generalized inverses goes back to Moore **[12]**. However, not until 1955 did the theory blossom, when Penrose **[13]** defined a uniquely determined generalized inverse for any matrix A. Today we use the name *pseudoinverse* or *Moore–Penrose inverse*. Penrose showed that, given any matrix A, there is one and only one matrix B satisfying the following four conditions:

$$ABA=A; \quad BAB=B; \quad (AB)^*=AB; \quad (BA)^*=BA.$$

We can write A^+ for the (unique) solution B to these four equations. Our first goal is to obtain A^+ by beginning with a full rank factorization of A. By the way, it is easy to see from uniqueness that $A^{++}=A$ for any A.

Suppose $A\in\mathbb{C}_r^{m\times n}$, with $r>0$, and suppose $A=FG$ is a full rank factorization of A. Then $F\in\mathbb{C}_r^{m\times r}$, $G\in\mathbb{C}_r^{r\times n}$, and $r=r(A)=r(F)=r(G)$. Now G has full row rank, so GG^* has full rank in $\mathbb{C}^{r\times r}$, and hence is invertible. Similarly, F has full column rank, so F^*F has full rank in $\mathbb{C}^{r\times r}$ and is therefore invertible. We now have our first main result.

THEOREM 1. *Let $A\in\mathbb{C}_r^{m\times n}$ with $r(A)>0$, and suppose $A=FG$ is a full rank factorization of A. Then*

(1) $F^+=(F^*F)^{-1}F^*$
(2) $F^+F=I_r$, *the r-by-r identity matrix*
(3) $G^+=G^*(GG^*)^{-1}$
(4) $GG^+=I_r$
(5) $A^+=G^+F^+$

Proof. Items (2) and (4) are trivial consequences of the definitions. The existence of F^+, G^+, and A^+ follows from the discussion preceding the theorem. Thus it suffices to show F^+, G^+, and A^+ satisfy their respective Moore–Penrose equations. This boils down to "symbol pushing"; we illustrate just a few calculations to suggest the flavor. For example, we need $GG^+G=G$, but

$$GG^+G=G\big(G^*(GG^*)^{-1}\big)G=(GG^*)(GG^*)^{-1}G=IG=G$$

where I is the identity matrix. The next calculation is similar:

$$(G^+G)^*=\big(G^*(GG^*)^{-1}G\big)^*=G^*(GG^*)^{-1*}G^{**}=G^*(GG^*)^{*-1}G$$
$$=G^*(GG^*)^{-1}G=G^+G.$$

The remaining arguments for F^+ and G^+ are similar. For A^+, we compute

$$AA^+A=AG^+F^+A=FGG^+F^+FG=FI_rI_rG=FG=A.$$

The remaining computations are similar; we leave them to the reader.

Equations (2) and (4) above are trivial, but they point out one advantage of computing with full rank factorizations: F^+ is a left inverse of F while G^+ is a right inverse for G.

We can now clean up a loose end. We noted earlier that full rank factorizations are not unique: if $A = FG$ is one full rank factorization and R is invertible of appropriate size, then $A = (FR)(R^{-1}G)$ is another full rank factorization. Does it get any worse than this? The next theorem says no.

THEOREM 2. *Every matrix A in $\mathbb{C}_r^{m\times n}$ with $r(A) > 0$ has infinitely many full rank factorizations. However, if $A = FG = F_1G_1$ are two full rank factorizations of A, then there exists an invertible matrix R in $\mathbb{C}^{r\times r}$ such that $F_1 = FR$ and $G_1 = R^{-1}G$. Moreover, $G_1^+ = (R^{-1}G)^+ = G^+R$ and $F_1^+ = (FR)^+ = R^{-1}F^+$.*

Proof. The first claim is now clear. Suppose $A = FG = F_1G_1$ are two full rank factorizations of A. Then $F_1^+F_1G_1 = F_1^+FG$; since $F_1^*F_1 = I_r$, we have $G_1 = (F_1^+F)G$. Note that F_1^+F is r-by-r and

$$r = r(G_1) = r((F_1^+F)G) \le r(F_1^+F) \le r,$$

so F_1^+F has full rank r; therefore F_1^+F is invertible. Similar reasoning shows that GG_1^+ is invertible. Let $S = F_1^+F$ and $GG_1^+ = R$. Then

$$SR = F_1^+FGG_1^+ = F_1^+AG_1^+ = F_1^+F_1G_1G_1^+ = I_r,$$

so $S = R^{-1}$. Therefore, $G_1 = SG = R^{-1}G$ and $F_1 = FGG_1^+ = FR$. To complete the proof, we calculate

$$(FR)^+ = ((FR)^*(FR))^{-1}(FR)^* = (R^*F^*FR)^{-1}R^*F^* = R^{-1}(F^*F)^{-1}R^{*-1}R^*F^*$$
$$= R^{-1}(F^*F)^{-1}F^* = R^{-1}F^+.$$

The computation to show $G_1^+ = G^+R$ is similar.

Example. As in the preceding example, let

$$A = \begin{bmatrix} 3 & 6 & 13 \\ 2 & 4 & 9 \\ 1 & 2 & 3 \end{bmatrix} = FG = \begin{bmatrix} 3 & 13 \\ 2 & 9 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using Theorem 1, we find

$$G^+ = \begin{bmatrix} 1/5 & 0 \\ 2/5 & 0 \\ 0 & 1 \end{bmatrix}; \quad F^+ = \begin{bmatrix} -3/26 & -11/13 & 79/26 \\ 1/13 & 3/13 & -9/13 \end{bmatrix}.$$

The pseudoinverse of A is the matrix product

$$A^+ = G^+F^+ = \begin{bmatrix} -3/130 & -11/65 & 79/130 \\ -3/65 & -22/65 & 79/65 \\ 1/13 & 3/13 & -9/13 \end{bmatrix}.$$

5. Four fundamental projections

Strang **[11]** has popularized a structural view of any matrix A in $\mathbb{R}^{m\times n}$; we adapt his approach to $\mathbb{C}^{m\times n}$. He assigns four subspaces to A: the column space of A, denoted $\mathscr{R}(A)$; the null space of A, denoted $\mathscr{N}(A)$; the column space of A^*, $\mathscr{R}(A^*)$; and the null space of A^*, $\mathscr{N}(A^*)$. With the help of a full rank factorization of A and the pseudoinverse, we can easily compute the orthogonal projections onto these subspaces.

A *projection* is a matrix P with $P^2 = P = P^*$; each subspace of $\mathbb{C}^n$ uniquely determines such a projection. The set of fixed vectors ($Px = x$) for P then coincides with the subspace. A standard method of computing projections begins by finding an orthonormal basis for the subspace. This is unnecessary, given a full rank factorization, since the Moore–Penrose equations imply

(i) $AA^+ = FF^+$ is the projection onto $\mathscr{R}(A)$;
(ii) $A^+A = G^+G$ is the projection onto $\mathscr{R}(A^*)$;
(iii) $I_m - AA^+ = I_m - FF^+$ is the projection onto $\mathscr{N}(A^*)$;
(iv) $I_n - A^+A = I_n - G^+G$ is the projection onto $\mathscr{N}(A)$.

Continuing our example from above,

$$G^+G = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad I - G^+G = \begin{bmatrix} 4/5 & -2/5 & 0 \\ -2/5 & 1/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are, respectively, the projections onto $\mathscr{R}(A^*)$ and $\mathscr{N}(A)$, while

$$FF^+ = \begin{bmatrix} 17/26 & 6/13 & 3/20 \\ 6/13 & 5/13 & -2/13 \\ 3/26 & -2/13 & 25/26 \end{bmatrix} \quad \text{and} \quad I - FF^+ = \begin{bmatrix} 9/26 & -6/13 & -3/26 \\ -6/13 & 8/13 & 2/13 \\ -3/26 & 2/13 & 1/26 \end{bmatrix}$$

are the projections onto $\mathscr{R}(A^*)$ and $\mathscr{N}(A^*)$, respectively.

6. Special full rank factorizations

Suppose $A \in \mathbb{C}_r^{m \times n}$, with $r > 0$. We have seen that one way to produce a full rank factorization of A is to use as the columns of F a basis for the column space of A. Suppose we pick an orthonormal basis (if a basis isn't orthonormal, the Gram–Schmidt process can be applied). Then we have $A = FG$, where $F^*F = I_r$ because the columns of F are orthonormal. It is easy to check that F^* satisfies the four Moore–Penrose equations, so $F^* = F^+$. Therefore, among the many full rank factorizations $A = FG$, we can always select one for which $F^+ = F^*$. We'll call such a full rank factorization *orthogonal*. A *unitary* matrix U is one for which $U^* = U^{-1}$. Since unitary matrices preserve length and orthogonality, we see that $A = (FU)(U^*G)$ is again orthogonal if $A = FG$ is. We summarize with a theorem.

THEOREM 3. *Every matrix* $A \in \mathbb{C}_r^{m \times n}$ *with* $r(A) > 0$ *has infinitely many orthogonal full rank factorizations.*

Next, we consider the special case of a projection matrix P in $\mathbb{C}_r^{n \times n}$. First we write P in an orthogonal full rank factorization $P = FG$, so $F^+ = F^*$; then

$$P = P^* = (FG)^* = G^*F^* = G^*F^+.$$

For a projection P, the Moore–Penrose equations imply that $P^+ = P^*$, so $P = P^+ = G^+F^+ = G^+F^*$, and $GP = GG^+F^* = F^*$. But then $P = PP = FGP = FF^*$. Actually, this must be the case since $FF^* = FF^+$ is the projection on the range of P, which is P. We have shown:

THEOREM 4. *Every projection* P *in* $\mathbb{C}_r^{n \times n}$ *has a full rank factorization* $P = FG$ *for which* $G = F^* = F^+$.

Returning to our ongoing example, the projection onto $\mathcal{N}(A^*)$ has the full rank factorization

$$I-FF^+=\begin{bmatrix} 9/26 & -6/13 & -3/26 \\ -6/13 & 8/13 & 2/13 \\ -3/26 & 2/13 & 1/26 \end{bmatrix}=\begin{bmatrix} 9/26 \\ -6/13 \\ -3/26 \end{bmatrix}\begin{bmatrix} 1 & -4/3 & -1/3 \end{bmatrix}.$$

This factorization is not orthogonal, but Gram–Schmidt is easy to apply. Normalizing the column vector gives $\begin{bmatrix} \frac{3}{\sqrt{26}} \\ -\frac{4}{\sqrt{26}} \\ -\frac{1}{\sqrt{26}} \end{bmatrix}$. Direct computation shows

$$I-FF^+=\begin{bmatrix} \frac{3}{\sqrt{26}} \\ -\frac{4}{\sqrt{26}} \\ -\frac{1}{\sqrt{26}} \end{bmatrix}\begin{bmatrix} \frac{3}{\sqrt{26}} & -\frac{4}{\sqrt{26}} & -\frac{1}{\sqrt{26}} \end{bmatrix}.$$

7. Matrix equivalence by full rank factorization

Matrices A and B are *equivalent*, and we write $A \sim B$, if B can be obtained from A by applying both elementary row and elementary column operations to A. Thus $A \sim B$ if there exist nonsingular matrices S and T with $SAT=B$.

Let $A=FG$ be a full rank factorization of A. We will show how F and G can be used to construct matrices S and T that will bring A into canonical form for the equivalence relation $\sim$. Consider

$$B=\begin{bmatrix} F^+ \\ \cdots\cdots\cdots \\ W_1(I-FF^+) \end{bmatrix} A\left[G^+ \vdots (I-G^+G)W_2\right]$$

where W_1 and W_2 are arbitrary matrices of appropriate dimension ($W_1 \in \mathbb{C}^{(m-r)\times m}$ and $W_2 \in \mathbb{C}^{n\times(n-r)}$). Computation gives

$$B=\begin{bmatrix} F^+AG^+ & F^+A(I-G^+G)W_2 \\ W_1(I-FF^+)AG^+ & A(I-GG^+)W_2 \end{bmatrix}=\begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}.$$

In fact, we could take *any* r-by-r matrix M and compute

$$\begin{bmatrix} F^+ \\ \cdots\cdots\cdots \\ W_1(I-FF^+) \end{bmatrix} A\left[G^+M \vdots (I-G^+G)W_2\right]=\begin{bmatrix} M & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix},$$

again with W_1 and W_2 arbitrary matrices of appropriate size.

We have uncovered the interesting fact that every full rank factorization of A leads to a *diagonal reduction* of A. Of course, the matrices S and T flanking A need not be invertible, but this can be arranged, again using full rank factorizations. Starting with any full rank factorization $A=FG$, we construct the projections

$$I-FF^+=F_1F_1^*=F_1F_1^+ \quad \text{and} \quad I-G^+G=F_2F_2^*=F_2F_2^+,$$

each in an orthogonal full rank factorization.

To make S invertible, we choose W_1 judiciously. A computation gives

$$\begin{bmatrix} F^+ \\ \cdots\cdots\cdots \\ W_1(I-FF^+) \end{bmatrix}\left[F \,\vdots\, (I-FF^+)W_1^*\right] = \begin{bmatrix} F^+F & F^+(I-FF^+)W_1^* \\ W_1(I-FF^+)F & W_1(I-FF^+)W_1^* \end{bmatrix}$$
$$= \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & W_1(I-FF^+)W_1^* \end{bmatrix}.$$

We need the identity matrix to appear in the lower right-hand corner of the preceding matrix. If we choose $W_1 = F_1^+ = F_1^*$, then

$$W_1(I-FF^+)W_1^* = W_1F_1F_1^*W_1^* = F_1^+F_1F_1^*F_1^{**} = IF_1^+F_1 = I.$$

Thus $S^{-1} = [F \,\vdots\, (I-FF^+)F_1] = [F \,\vdots\, F_1]$. Similarly, for $T = [G^+ \,\vdots\, (I-G^+G)W_2]$ we choose $W_2 = F_2$ and find that

$$T^{-1} = \begin{bmatrix} G \\ F_2^*(I-G^+G) \end{bmatrix} = \begin{bmatrix} G \\ F_2^+ \end{bmatrix}.$$

So we have derived a familiar theorem but with a new twist.

THEOREM 5. *Every matrix A in $\mathbb{C}_r^{m\times n}$ is equivalent to* $\begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}$. *If $A = FG$ is a full rank factorization of A and $I - FF^+ = F_1F_1^+$ and $I - G^+G = F_2F_2^+$ are orthogonal full rank factorizations, then*

$$S = \begin{bmatrix} F^+ \\ \cdots \\ F_1^+ \end{bmatrix} \quad \text{and} \quad T = \left[G^+ \,\vdots\, F_2\right]$$

are invertible, and

$$SAT = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}.$$

A little more is true: the matrix S can be chosen to be *unitary*, not just invertible. To see this, we begin with an orthogonal full rank factorization $A = FG$, where $F^* = F^+$. Then

$$SS^* = \begin{bmatrix} F^+ \\ \cdots\cdots\cdots \\ W_1(I-FF^+) \end{bmatrix}\left[(F^+)^* \,\vdots\, (I-FF^+)W_1^*\right]$$
$$= \begin{bmatrix} F^+(F^+)^* & \mathbb{O} \\ \mathbb{O} & W_1(I-FF^+)W_1^* \end{bmatrix} = \begin{bmatrix} F^+F & \mathbb{O} \\ \mathbb{O} & W_1(I-FF^+)W_1^* \end{bmatrix}$$
$$= \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & W_1(I-FF^+)W_1^* \end{bmatrix}.$$

Now we choose $W_1 = F_1^+$ as before to get $SS^* = I$.

8. The singular value decomposition

We will now see how to derive the singular value decomposition of a matrix beginning with a full rank factorization. We have seen that, for given $A \in \mathbb{C}_r^{m\times n}$, full rank factorizations lead to explicit S and T, with S unitary and T invertible, such that

$SAT = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}$. Can we get T unitary as well? The answer is yes if we replace the matrix I_r with a slightly more general diagonal matrix D.

As before, we begin with an orthogonal full rank factorization $A = FG$ with $F^+ = F^*$. We also use the factorizations $I - FF^+ = F_1F_1^+ = F_1F_1^*$ and $I - G^+G = F_2F_2^* = F_2F_2^+$. Then

$$\begin{bmatrix} F^+ \\ F_1^+(I - FF^+) \end{bmatrix} A\left[G^+D \vdots (I - GG^+)W_2\right] = \begin{bmatrix} D & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}.$$

The matrix $\begin{bmatrix} F^+ \\ F_1^+ \end{bmatrix}$ is unitary; let's call it U^*; the matrix W_2 is arbitrary. Next we consider $V = \left[G^+D \vdots (I - G^+G)W_2\right]$, which we would like to make unitary by choice of D and W_2. But

$$\begin{aligned} V^*V &= \begin{bmatrix} (G^+D)^* \\ [(I - G^+G)W_2]^* \end{bmatrix}\left[G^+D \vdots (I - G^+G)W_2\right] \\ &= \begin{bmatrix} D^*G^{+*}G^+D & D^*G^{+*}(I - G^+G)W_2 \\ W_2^*(I - G^+G)G^+D & W_2^*(I - G^+G)W_2 \end{bmatrix} \\ &= \begin{bmatrix} D^*G^{+*}G^+D & \mathbb{O} \\ \mathbb{O} & W_2^*(I - G^+G)W_2 \end{bmatrix} \end{aligned}$$

since $G^{+*} = (G^+GG^+)^* = G^{+*}(G^+G)^* = G^{+*}G^+G$ implies that $D^*G^{+*}(I - G^+G)W_2 = \mathbb{O}$. We have already seen that with $W_2 = F_2^+$ we get I_{m-r} in the lower right position. So the problem reduces to solving $D^*G^{+*}G^+D = I_r$ for a suitable D. But $G = F^+A = F^*A$, so

$$D^*G^{+*}G^+D = D^*(GG^*)^+D = D^*(G^{+*})^{-1} = D^*(F^*AA^*F)^{-1}D.$$

To achieve the identity matrix I_r we need $F^*AA^*F = DD^*$ or, equivalently, $AA^*F = FDD^*$. We summarize our results in a theorem.

THEOREM 6. *Let $A = FG$ be an orthogonal full rank factorization. If there exists $D \in \mathbb{C}^{r\times r}$ with $GG^* = DD^*$, then there exist unitary matrices S and T with* $\mathrm{SAT} = \begin{bmatrix} D & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}$.

One way to exhibit such a matrix D for a given A is to choose for the columns of F an orthonormal basis consisting of the eigenvectors of AA^* corresponding to nonzero eigenvalues. Since AA^* is positive semidefinite we know its eigenvalues are non-negative. Then $AA^*F = FE$ where E is the r-by-r diagonal matrix of real positive eigenvalues of AA^*. Let D be the diagonal matrix of the positive square roots of these eigenvalues. Then

$$D^*[F^*AA^*F]^{-1}D = DE^{-1}D = I_r.$$

We have captured the classical singular value decomposition in the context of full rank factorization.

THEOREM 7. *Let $A = FG$ be a full rank factorization of A where the columns of F are an orthonormal basis consisting of eigenvectors of AA^* corresponding to nonzero eigenvalues. Suppose*

$$I - FF^+ = F_1F_1^* = F_1F_1^+ \quad \text{and} \quad I - G^*G = F_2F_2^* = F_2F_2^+.$$

Then there exist unitary matrices U and V with $U^*AV = \begin{bmatrix} D_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}$, *where* D_r *is an r-by-r diagonal matrix whose diagonal entries are the square roots of the nonzero eigenvalues of* AA^*. *The matrices U and V can be constructed explicitly from* $U^* = \begin{bmatrix} F^* \\ F_1^+ \end{bmatrix}$ *and* $V = \left[G^+D \vdots F_2 \right]$.

Conclusion

We have only scratched the surface of what can be obtained from full rank factorizations. More information can be found in the references. Of course, ours is not the only point of view. One could begin with the classical singular value decomposition and derive a full rank factorization from it. Briefly, it goes like this: We write

$$A = U\begin{bmatrix} E_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}V^* = \left[U_r \vdots U_{m-r} \right]\begin{bmatrix} E_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}\begin{bmatrix} V_r \\ \cdots\cdots \\ V_{n-r} \end{bmatrix}^* = U_r E_r V_r^* = U_r(E\, V_r^*).$$

Taking $F = U_r$ and $G = EV_r^*$ we get a full rank (in fact, orthogonal full rank) factorization of A in $\mathbb{C}_r^{m\times n}$. It can be shown that A^+ has full rank factorization $A^+ = V_r(E^{-1}U_r^*)$, $V_rV_r^*$ is the projection on $\mathscr{R}(A^*)$, $V_{n-r}V_{n-r}^*$ is the projection on the $\mathscr{N}(A)$, $U_rU_r^*$ is the projection on $\mathscr{R}(A)$, and $U_{m-r}U_{m-r}^*$ is the projection on $\mathscr{N}(A^*)$. From a numerical linear algebra point of view, starting with the singular value decomposition probably makes more sense since there are effective and stable algorithms available for its direct computation. Also, there are full rank QR and full rank LU factorizations and ways to produce bases for the fundamental subspaces of a matrix. But that is another story.

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