Compactness

XI

In this chapter, we consider spaces having a strengthened version of the Lindelöf property; such spaces play an important role in all branches of mathematics.

1. Compact Spaces

1.1 Definition A Hausdorff space Y is compact if each open covering has a finite subcovering.

Ex. 1 A discrete space is compact if and only if it is finite. The proof in VIII, **2**, Ex. 2, shows the ordinal space $[0, \Omega]$ is compact; note that a compact space need not be even 1° countable.

Ex. 2 In any space X, all finite subsets, and $\varnothing$, are compact subsets. If $\varphi: Z^+ \to X$ is a sequence convergent to x_0, then $A = x_0 \cup \varphi(Z^+)$ is a compact subset of X: any set of an open covering of A that contains x_0 contains all but at most finitely many elements of A. Observe that an infinite subset of A that does not contain x_0 is not a compact set.

Ex. 3 E^1 is not compact, since the open covering $]-n, n[$, $n = 1, 2, \cdots$ has no finite subcovering. However, each closed finite interval $[a, b]$ is compact: In fact, given any open covering $\{U_\alpha\}$ of $[a, b]$, let $c = \sup\{x \mid [a, x]$ can be covered by finitely many $U_\alpha\}$; if $c < b$, we derive a contradiction to the definition of c by choosing any $U_\alpha \supset c$, observing that there is a $B(c, r) \subset U_\alpha$, and that since $[a, c - (r/2)]$ can be covered by finitely many sets $U_{\alpha_1}, \cdots, U_{\alpha_n}$, these sets together with U_α are a finite open covering of $[a, c + (r/2)]$.